

**University of Boumerdes  
Institute of Electricity and Electronics  
DEPARTMENT OF RESEARCH**

# **THESIS**

Presented in partial fulfilment of the  
**DEGREE OF MAGISTER**

in  
**Electronic Systems Engineering**

by  
**HOURIA AMROUNE**

**MULTIVARIABLE DESIGN VIA THE  
SOLUTION OF THE LYAPUNOV EQUATION**

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## **Dedication**

I dedicate this modest work to the memory of my parents, especially my mother, without whom I would have never reached this stage. To my brothers and sisters for their patience and support.

To Mike and to all my friends wherever they are.

To the families Harzallah and Medar.

# Acknowledgements

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## Abstract

Relocating poles in control theory has been the subject of an important number of research studies. Using state feedback for this purpose is a widely used technique because of its simplicity and the quality of control achieved. The control signal is designed to be a function of the state and the reference signal.

Controllability of an open-loop system is equivalent to the possibility of assigning an arbitrary set of eigenvalues to the closed-loop system's matrix by means of suitable linear feedback of the state.

Consider a MIMO system described by the following equation:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

where  $A$ ,  $B$  and  $C$  are respectively  $n \times n$ ,  $n \times p$  and  $q \times n$  constant real matrices,  $x(t) \in \mathcal{R}^n$ ,  $u(t) \in \mathcal{R}^p$ .

The state feedback control design requires finding a control signal of the form  $u(t) = -Kx(t)$  such that the designed system matrix  $(A - BK)$  has a set of desired poles.

Several methods have been proposed for the computation of the feedback gain matrix  $K$ , [6, 14, 33].

In this work, Chen's algorithm [6] will be used to compute the matrix  $K$ ; this algorithm requires choosing an arbitrary  $n \times n$  matrix  $F$  whose eigenvalues (not common to those of  $A$ ) are the desired poles of the system, and an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  is observable.

Making  $(A - BK)$  similar to  $F$  leads to a matrix equation of the form

$$AT - TF = B\bar{K}$$

which is of the Lyapunov type.

The solution of the above control problem requires finding a nonsingular solution  $T$  to deduce

$$K = \bar{K}T^{-1}$$

In the past few years, several numerical methods have been suggested for the solution of the Lyapunov type equation, in general [4, 18, 28, 45, 46]. Among these methods, the Hessenberg Schur method appeared to be the most efficient numerically [4, 6, 18]. It has therefore been chosen to solve our matrix equation.

Our contribution consists in the suggestion of suitable forms of  $F$  and  $\bar{K}$  to ensure the nonsingularity of the solution  $T$  while keeping the computational procedure as efficient as possible. It is proposed in this work to assign a Jordan or Controller form to  $F$  and apply the Hessenberg Schur method. The proposed approach is further extended to the block matrix case.

## TABLE OF CONTENTS

Acknowledgement.....	ii
Abstract.....	iii
Chapter 1. INTRODUCTION.....	01
1.1 Introduction.....	01
1.2 Problem statement.....	02
1.3 Description of the Thesis.....	03
Chapter 2. MATHEMATICAL PRELIMINARIES.....	05
2.1 Introduction.....	05
2.2 Similarity Transformation.....	05
2.3 Householder Transformations.....	06
2.4 Reduction to Hessenberg Form.....	07
2.5 QR Algorithm.....	09
2.6 Gaussian Elimination.....	11
Chapter 3. SYSTEMS THEORY.....	14
3.1 Introduction.....	14
3.2 System Descriptions.....	14
3.3 Controllability of Linear Systems.....	16
3.4 Observability of Linear Systems.....	18
3.5 Canonical Forms.....	20
3.6 Linear State Feedback.....	32

Chapter 4.	SOLUTION OF THE LYAPUNOV EQUATION.....	44
	4.1 Introduction.....	44
	4.2 Solution Given by Integration.....	44
	4.3 Solution via the Kronecker Product.....	46
	4.4 Bartels-Stewart Algorithm.....	49
	4.5 Hessenberg-Schur Algorithm.....	50
Chapter 5.	PROPOSED APPROACH.....	52
	5.1 Introduction.....	52
	5.2 The Proposed Algorithm.....	52
	5.3 Observability of $(F, \bar{K})$ .....	54
	5.4 Jordan Form Approach.....	61
	5.5 Controller Form Approach.....	63
	5.6 The Proposed Block Form Algorithm.....	66
Chapter 6.	IMPLEMENTATION AND NUMERICAL RESULTS.....	69
	6.1 Introduction.....	69
	6.2 Algorithms.....	69
	6.3 Perturbation Analysis.....	73
	6.4 Numerical Implementation.....	75
Chapter 7	CONCLUSION.....	95
	References.....	98

**1.1 Introduction**

The study of a physical system can be carried out by applying to it various signals and measuring its responses. The performance can then be improved by adjusting some of its parameters or connecting to it some compensator. This method is said to be empirical.

If the specifications on the performance become very precise and stringent, the empirical method may become unsatisfactory or inadequate if physical systems become very complicated, too expensive or too dangerous to be experimented. In these cases, analytical methods become indispensable and this study can be decomposed into four parts: modeling or development of mathematical equation description, analysis, design and testing. Once a mathematical description of a system is obtained, quantitative and qualitative analyses can be done.

Time domain methods were used to tackle the problems of automatic control which were, at first, treated as problems of rational mechanics and soon came, particularly under the influence of electrical and electronic engineers to be treated as filtering problems, i.e., using frequential methods.

Although the frequency domain techniques, associated with the concept of a transfer function, have yielded the single variable methods, they began gradually to give place to a theory of control in state-space. The increasing complexity of systems to be controlled led to inexplicable errors when these methods were used; especially in multivariable cases,

such errors are most of the time due to the lack of appreciation of the ideas of controllability and observability.

Relocating the eigenvalues of a given system using the feedback of the state variables is among the first applications of the state-space methods to linear systems. This is a widely used technique; this popularity is due to the fact of being a well established approach compared to output feedback for example, and to its simplicity since the control signal is designed to be a function of the state and the reference signal. According to Kalman et al., J. Bertram in 1959 was perhaps the first to realize that if a given system realization was state controllable, then by state-variable feedback, any desired characteristic polynomial could be obtained.

## 1.2 Problem Statement

We are given a realization

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{1.1}$$

We wish to modify the given system by the use of state-variable feedback so as to obtain a new system with specified eigenvalues.

State-variable feedback is obtained by the substitution  $u = -Kx$ . After feedback, the following realization is obtained

$$\begin{aligned}\dot{x} &= (A - BK)x \\ y &= Cx\end{aligned}\tag{1.2}$$

whose poles are given by the eigenvalues of the closed-loop system matrix  $(A - BK)$ . The state feedback design problem is then summarized into finding a suitable and efficient way to compute the feedback matrix  $K$ .

Unlike some other methods, Chen's algorithm allows the designer to preset the form of the closed-loop system matrix and therefore preset the closed-loop eigenstructure.

This algorithm consists then of choosing an arbitrary  $n \times n$  matrix  $F$  whose eigenvalues are the desired poles of the compensated system, and an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $F$  has no eigenvalues in common with those of  $A$ , and the couple  $(F, \bar{K})$  is observable. Saying that  $(A - BK)$  has the same set of eigenvalues as  $F$  means that these two matrices are similar; consequently, there exists a similarity transformation matrix  $T$  which has to be determined. The above statements result in the following equation:

$$AT - TF = B\bar{K} \quad (1.3)$$

to be solved for  $T$ .

If  $T$  is nonsingular, then the solution is given by  $K = \bar{K}T^{-1}$ ; if  $T$  is singular then a different  $F$  and/or a different  $\bar{K}$  have to be chosen and the process is repeated.

This equation has the form of a well known equation in matrix theory: the Lyapunov equation.

From what has just been said, it is clear that this method requires the solution of the Lyapunov equation. This equation has been the subject of many research papers in numerical methods. In this thesis, it is question first of selecting an effective method to solve this equation, then selecting accordingly, suitable forms of  $F$  and  $\bar{K}$  for a best optimization of the algorithm. By best optimization we mean minimum cost by reducing the time and space complexities of each step and also ensuring a solution is obtained at first run of the algorithm.

### 1.3 Description of the Thesis

We will now give a brief description of the thesis.

Chapter one gives a brief summary of the problem to be tackled in this thesis.

Chapter two covers some necessary mathematical preliminaries needed in the development of the present work.

Chapter three gives a brief background about systems theory, and covers systems descriptions, controllability and observability required in the study of every control system and finally some canonical forms.

Chapter four presents some numerical methods found in the literature for the solution of the Lyapunov equation.

Chapter five consists in the proposed approaches for the solution of the problem; we will generalize the algorithm to handle block matrices.

Chapter six presents an analysis to effectively compare the different approaches, this analysis will include the effect of the forms chosen as well as the impact of the number of blocks on the computations. In this chapter, some numerical examples will be presented to illustrate and test the results of the algorithm.

Finally, chapter seven concludes the thesis.

## 2.1 Introduction

Matrix theory plays a great role in the study of control systems, especially in multivariable cases. With the help of numerical methods, this study has been simplified enabling us to transform systems from general forms to special forms suitable for computer analysis without changing their spectral characteristics. Within this scope, we introduce here some transformations required in our design.

## 2.2 Similarity Transformation

The reduction of a matrix of general form to one of canonical form will usually be achieved progressively by performing a sequence of simple similarity transformations.

Two square matrices  $A$  and  $\bar{A}$  are said to be similar if there exists a nonsingular matrix  $P$  satisfying the following equalities:

$$\begin{aligned}\bar{A} &= PAP^{-1} = Q^{-1}AQ \\ A &= P^{-1}\bar{A}P = Q\bar{A}Q^{-1}\end{aligned}\tag{2.1}$$

The transformation defined above is called a similarity transformation.

**Theorem 2.1:**

- (a) Similar matrices have the same characteristic polynomial and the same eigenvalues.
- (b) Suppose that  $B$  is similar to  $A$  with  $B = P^{-1}AP$ . Then  $x$  is an eigenvector of  $A$  associated with the eigenvalue  $\lambda$  if and only if  $P^{-1}x$  is an eigenvector of  $B$  associated with the eigenvalue  $\lambda$ .

**Proof:** (see [35])

**Note:** The eigenvalues and eigenvectors of a matrix are not preserved when the matrix is premultiplied by another matrix such as elementary row operations (interchanging rows of a matrix may change eigenvalues).

### 2.3 Householder Transformation

Householder (1958) was the first to use elementary reflectors in a systematic way to introduce zeros into a matrix. Elementary reflectors are also known as elementary Hermitian matrices and as Householder transformations.

**Definition 2.1:**

For nonzero  $w$  in  $\mathfrak{R}^n$ , the  $n \times n$  Householder matrix  $H_w$  is defined as:

$$H_w = I_n - \left( \frac{2}{w^T w} \right) w w^T \quad (2.2)$$

**Theorem 2.2:**

Let  $H_w$  be the Householder matrix defined by a nonzero  $w$  in  $\mathfrak{R}^n$ . Then:

- $H_w$  is symmetric ( $H_w^T = H_w$ )

- $H_w$  is orthogonal ( $H_w^T H_w = I$ )
- $H_w$  is involutory ( $H_w^2 = I$ )
- For each  $x$ ,  $H_w x$  equals the reflection of  $x$  about the subspace of all  $v$  orthogonal to  $w$ .
- $\text{Det } H_w = -1$
- For any nonzero  $x$  and nonzero  $y$  in  $\mathfrak{R}^n$  with  $x \neq y$ , there exists an  $H_w$  such that  $H_w x = \alpha y$  for some real number  $\alpha$ .

1. Unless  $x$  equals a positive multiple of  $y$ ,  $w$  can be taken as

$$w = x - \left( \frac{\|x\|_2}{\|y\|_2} \right) y \text{ and then } H_w x = \left( \frac{\|x\|_2}{\|y\|_2} \right) y$$

2. Unless  $x$  equals a negative multiple of  $y$ ,  $w$  can be taken as

$$w = x + \left( \frac{\|x\|_2}{\|y\|_2} \right) y \text{ and then } H_w x = - \left( \frac{\|x\|_2}{\|y\|_2} \right) y$$

**Proof:** (see [35, 44])

## 2.4 Reduction to Hessenberg Form

### Theorem 2.3

$A$  is a  $n \times n$  real matrix. There is an easily computable sequence of at most  $n-2$  Householder matrices  $H_1, \dots, H_{n-2}$  implementing an orthogonal similarity

$$(H_{n-2} \dots H_2 H_1) A (H_1 H_2 \dots H_{n-2}) = H \quad (2.3)$$

such that the transformed matrix  $H$  is in (upper) Hessenberg form defined as follows:

$H$  is almost upper-triangular in that only its first subdiagonal can contain nonzero entries, that is

$$\langle H \rangle_{ij} = 0 \quad \text{for } i \geq j + 2.$$

**Proof:** (see [35])

A typical step of the reduction to Hessenberg form goes as follows: at the second step an order six matrix will have the form

$$A_2 = \begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & \bar{x} & x & x & x & x \\ 0 & \bar{x} & x & x & x & x \\ 0 & \bar{x} & x & x & x & x \end{bmatrix}$$

An elementary reflector  $H_2$  of the form  $H_2 = \begin{pmatrix} I_2 & 0 \\ 0 & R_2 \end{pmatrix}$  is chosen so that  $H_2 A_2$  has zeros in the position distinguished in  $A_2$ . Obviously, the zero elements in the first column of  $A_2$  are not disturbed by the premultiplication by  $H_2$ . Furthermore, postmultiplication of a matrix by  $H_2^H$  leaves its first two columns undisturbed. So  $A_3 = H_2 A_2 H_2^H$  has the form

$$A_3 = \begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \end{bmatrix}$$

which carries the reduction one step further.

In general at the  $k$ th step, the reduced matrix  $A_k$  will have the form:

$$A_k = \begin{pmatrix} A_{11}^{(k)} & a_{12}^{(k)} & A_{13}^{(k)} \\ 0 & a_{22}^{(k)} & A_{23}^{(k)} \end{pmatrix}$$

where  $A_{11}^{(k)} \in \mathfrak{R}^{k \times (k-1)}$ . Let  $R_k \in \mathfrak{R}^{(n-k) \times (n-k)}$  be an elementary reflector such that

$$R_k a_{22}^{(k)} = \pm \|a_{22}^{(k)}\|_2 e_1$$

and let

$$U_k = \begin{pmatrix} I_k & 0 \\ 0 & R_k \end{pmatrix}$$

Then

$$A_{k+1} = U_k A_k U_k^T \begin{pmatrix} A_{11}^{(k)} & a_{12}^{(k)} & A_{13}^{(k)} R_k \\ 0 & \pm \|a_{22}^{(k)}\|_2 e_1 & R_k A_{23}^{(k)} R_k \end{pmatrix}$$

which carries the reduction one step further.

## 2.5 QR Algorithm (Reduction to Schur Form)

QR algorithm (Francis 1961) is an iterative technique and is the heart of the methods for reducing a real matrix to quasi-triangular form by orthogonal similarity transformations. In this algorithm, a factorization into the product of a unitary matrix  $Q$  and an upper-triangular matrix  $R$  is used instead of triangular decomposition.

### 2.5.1 QR Decomposition

The QR factorization of a matrix  $A$  is done by reducing it to triangular form by means of premultiplication by elementary reflectors.

**Theorem 2.4:**

Let  $A$  be  $n \times m$  real matrix. A sequence  $H_1 \dots H_m$  of at most  $m$  Householder matrices can be easily computed so that

$$H_m H_{m-1} \dots H_1 A = R \quad (2.4)$$

where  $R$  is upper-triangular and has nonnegative entries on the main diagonal; equivalently:

$$A = QR \quad (2.5)$$

where  $Q = H_1 H_2 \dots H_m$  is  $n \times n$  and orthogonal.

**Proof:** (see [35])

Every  $n \times n$  orthogonal matrix  $Q$  can be written as the product  $Q = H_1 H_2 \dots H_n$  of at most  $n$  Householder matrices.

**2.5.2 Outline of the basic QR Algorithm**

1. Given the real  $n \times n$  matrix  $A$ , use  $n-2$  Householder matrices to compute the orthogonally similar matrix  $H$  in Hessenberg form. Define  $A_1 = H$  and set  $i=1$ .
2. Use  $n$  Householder matrices to compute a QR-decomposition of  $A_i$  as  $A_i = Q_i R_i$  where  $Q_i$  is orthogonal and  $R_i$  is upper-triangular.
3. Define  $A_{i+1} = R_i Q_i$ . If  $A_{i+1}$  has converged to a Schur form then stop; otherwise, increase  $i$  by 1 and return to step 2.

The process of computing a similarity transformation is usually expensive. For instance, if  $A$  and  $P$  are real  $n \times n$  matrices with  $P$  orthogonal, it requires about  $2n^3$  multiplications to compute the matrix  $\bar{A} = PAP^H$ . If an iterative process requires even as little as one similarity transformation per eigenvalue, the complete reduction will require about  $2n^4$  multiplications as opposed to, say,  $\frac{2}{3}n^3$  multiplications for the Gaussian reduction.

One cure for this problem is to transform the given matrix to a simple form which has a large number of zero elements. Since such a processing algorithm should be direct, the simple form will have to be more complex than a triangular matrix. The transformation to Hessenberg form by similarity transformations is the most suitable one. The algorithm may be carried out using elementary reflectors and requires about  $\frac{5}{3}n^3$  multiplications.

## 2.6 Gaussian Elimination

The determination of a simple nonsingular matrix  $P$  such that  $PA$  is upper triangular is of fundamental importance since it leads to a direct method for solving the matrix equation of the form:

$$Ax = b \tag{2.6}$$

where  $A$  is an  $n \times n$  matrix.

If  $x$  is a solution of (2.6), then it is also a solution of

$$PAx = Pb \tag{2.7}$$

where  $P$  is any  $m \times n$  matrix.

The triangularization of  $A$  using Gaussian Elimination consists of  $(n - 1)$  major steps.

We denote the initial set of equations by:

$$A_0 x = b_0. \quad (2.8)$$

Each step leads to the production of a new set of equations equivalent to the original set.

The  $p$ th equivalent set is denoted by:

$$A_p x = b_p. \quad (2.9)$$

with  $A_p$  is upper triangular as far as its first  $p$  columns are concerned, so that typically for  $n = 5$ ,  $p = 3$ , the equations (2.9) have the form

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

the crosses denote elements which are, in general, nonzero. These steps are summarized in the following algorithm.

**The algorithm:**

Take for the matrix  $P_r$  the elementary matrix  $N_r$  with

$$n_{ir} = \frac{a_{ir}^{(r-1)}}{a_{rr}^{(r-1)}}. \quad (2.10)$$

The divisor  $a_{rr}^{(r-1)}$  is called the  $r$ th pivot.

Premultiplication by  $N_r$  results in the subtraction of a multiple  $n_{ir}$  of the  $r$ th row from the  $i$ th row for each of the values of  $i$  from  $(r+1)$  to  $n$ , the multiples being chosen so as to eliminate the last  $(n-r)$  elements in column  $r$ . The same operations are performed on the right hand side  $b_{r-1}$ . Rows 1 to  $(r-1)$  are unaltered. Row  $r$  is also unaltered, in effect the  $r$ th equation is used to eliminate  $x_r$  from equations  $(r+1)$  to  $n$  of  $A_{r-1}x = b_{r-1}$ .

### 3.1 Introduction

The very first step in the analytical study of a physical system is the development of mathematical equations to describe the system. We can encounter many possible representations but these are mainly grouped into two types of equations: input-output description and state-variable description. This is what we will have a look at in the next section. We will then see two qualitative properties of dynamical equations: the controllability and observability in the following two sections. Linear state feedback will be discussed in the fifth section and finally some canonical forms will be presented.

### 3.2 System Descriptions

#### 3.2.1 The input-Output Description:

This description considers only the terminal properties of a system; it gives a mathematical relation between the inputs and the outputs of the system. In this case the system is considered as a "black box" which can be accessed only by the input and output terminals.

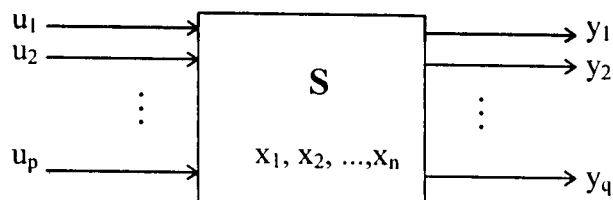


Figure 3.1 General system with p inputs and q outputs

When the resulting equation is linear with constant coefficients, Laplace or Z-transforms can be used to define the input-output transfer function. In the case of multivariable systems, matrix notation and transfer matrices are used in the description.

### 3.2.2 The State-Variable Description:

The equations that describe the internal and terminal behavior of the system are called the internal or state-variable description of the system. This set of equations is as follows:

$$\begin{aligned}\dot{x}(t) &= h(x(t), u(t), t) && \text{(State equation)} \\ y(t) &= g(x(t), t) && \text{(Output equation)}\end{aligned}$$

and is called the dynamical equation. For the linear time invariant systems, these equations have the form below:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{3.1}$$

where  $A$ ,  $B$  and  $C$  are respectively  $n \times n$ ,  $n \times p$  and  $q \times n$  matrices. This equation will be used in the remainder of the thesis.

System analysis generally consists of a quantitative part and a qualitative part. The quantitative study deals with the exact response of the system to certain input and initial conditions.

The qualitative study concerns the general properties of a system. The following two sections will introduce two properties of linear dynamical equations which are very useful in the design of state feedback: Controllability and Observability.

### 3.3 Controllability of Linear Systems

Consider a linear continuous time-invariant system described by the state equations:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{3.2}$$

The linear system described by equation (3.2) is controllable at  $t_0$  if it is possible to find an input  $u(t)$  defined over  $t \in T$ , which will transfer the initial state  $x(t_0)$  to the origin at some finite time  $t_1 \in T, t_1 > t_0$ . That is, there exists some input  $u_{[t_0, t_1]}$  which gives  $x(t_1) = 0$  at a finite  $t_1 \in T$ . If this is true for all initial times  $t_0$  and all initial states  $x(t_0)$ , the system is completely state controllable.

Equivalently, the condition for controllability is that there exists a finite  $t_1$  on  $[t_0, t_1]$  such that all the rows of  $\phi(t_0, t)B(t)$  are linearly independent where in the case of linear time invariant systems,  $\phi(t_0, t)B(t) = e^{A(t_0 - t)}B$ . All elements of  $e^{A(t_0 - t)}B$  are linear combination of terms of the form  $t^k e^{\lambda t}$ ; hence, they are analytic on  $[0, \infty)$ , and consequently, the rows are linearly independent on  $[0, \infty)$ , they are independent on  $[t_0, t_1]$  for any  $t_0$  and any  $t_1 > t_0$ .

**Theorem 3.1:**

The n-dimensional linear time-invariant state equation (3.2) is controllable if and only if the following equivalent conditions are satisfied:

1. All rows of  $e^{-At}B$  are linearly independent on  $[0, \infty)$  over  $C$ , or all rows of  $(sI - A)^{-1}B$  are linearly independent over  $C$ .

## 2. The controllability Grammian

$$W_{ct} = \int_0^t e^{A\tau} B B^* e^{A^*\tau} d\tau$$

is nonsingular for any  $t > 0$ .

3. The  $n \times np$  controllability matrix

$$U_c = [B \ AB \ A^2 B \ \dots \ A^{n-1} B]$$

has rank  $n$ .

**Proof:** (see [6]).

**Remark:** The rank defect of  $U_c$  indicates the number of modes of the system which cannot be influenced by the inputs  $u$ . However, it does not tell us which modes are concerned. Other tests based on eigenprojector theory are able to indicate which modes are not controllable.

**Corollary 3.1:**

The state equation (3.2) is controllable if and only if the  $n \times (n - \hat{p} + 1)p$  matrix

$$\hat{U}_c = [B \ AB \ A^2 B \ \dots \ A^{n-\hat{p}} B] \quad (3.3)$$

where  $\hat{p}$  is the rank of  $B$ , has rank  $n$ .

**Proof:** (see [6]).

**Controllability Indices**

Consider a system  $[A, B, C]$ , if this system is controllable, then the matrix  $U_c$  defined as:

$$U_c = [B \ AB \ A^2 B \ \dots \ A^{n-1} B] \quad (3.4)$$

has rank  $n$ . Thus a nonsingular  $n \times n$  transformation  $P$  can be formed from the columns of  $U_c$ .

The total number of columns in  $U_c$  is  $np$ ; hence, there are many possible ways of choosing  $n$  linearly independent columns.

One way of choosing these columns is as follows:

Inspect the columns of  $U_c$  from left to right and retain only those vectors which are linearly independent of those previously selected.

Arrange the  $n$  linearly independent vectors so selected to construct a matrix  $\Gamma_c$ , where

$$\Gamma_c = \left[ b_1, Ab_1, \dots, A^{r_1-1}b_1, b_2, Ab_2, \dots, A^{r_2-1}b_2, \dots, A^{r_p-1}b_p \right]. \quad (3.5)$$

The integers  $(r_1, \dots, r_p)$  are known as the **controllability indices** of the system, where  $r_i$  is the number of linearly independent vectors associated with  $b_i$ , and  $\sum_{i=1}^p r_i = n$  if the system is controllable.

The parameter

$$r_c = \max_i r_i; \quad i = 1, \dots, p \quad (3.6)$$

is called the controllability index for the system.

### 3.4 Observability of Linear Systems

Consider the  $n$ -dimensional linear time-invariant state equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (3.7)$$

The time interval of interest is  $[0, \infty)$ . If a dynamical equation is observable it is observable at any time on the interval.

**Theorem 3.2:**

The  $n$ -dimensional linear time-invariant dynamical equation (3.7) is observable if and only if any of the following equivalent conditions is satisfied:

1. All columns of  $Ce^{At}$  are linearly independent on  $[0, \infty)$  over  $C$ , or all columns of  $C(sI - A)^{-1}$  are linearly independent over  $C$

2. The observability Grammian

$$W_{ot} = \int_0^t e^{A^* \tau} C^* C e^{A \tau} d\tau$$

is nonsingular for any  $t > 0$ .

3. The  $n \times n$  observability matrix

$$U_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix} \quad (3.8)$$

has rank  $n$ .

4. For every eigenvalue  $\lambda$  of  $A$ , the  $(n+q) \times n$  complex matrix

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

has rank  $n$ , or equivalently  $(sI - A)$  and  $C$  are right coprime.

**Proof:** (see [6]).

### Observability Indices

Consider a system  $[A, B, C]$ , if this system is observable, then the matrix  $U_o$  defined by the matrix in (3.8) has rank  $n$ , which means there is a possibility of extracting  $n$  independent columns from  $U_o$ .

How to choose these columns?

By duality with controllability, these columns can be selected by inspecting the columns of  $U_o$  from left to right and retain only those vectors which are linearly independent of those previously selected.

Arrange the  $n$  linearly independent vectors so selected to form a new matrix  $\Gamma_o$  where

$$\Gamma_o = \left[ c_1^T, A^T c_1^T, \dots, (A^T)^{r_1-1} c_1^T, c_2^T, A^T c_2^T, \dots, (A^T)^{r_2-1} c_2^T, \dots, c_q^T, A^T c_q^T, \dots, (A^T)^{r_q-1} c_q^T \right] \quad (3.9)$$

in this case, the corresponding integers  $(r_1, \dots, r_q)$  are called the **observability indices** of the

system, and  $\sum_{i=1}^q r_i = n$  if the system is observable.

The parameter

$$r_o = \max_i r_i; \quad i = 1, \dots, q \quad (3.10)$$

is called the observability index for the system.

### 3.5 Canonical Forms

Canonical forms of the state-variable model are fully equivalent, each of which has certain practical advantages when applied to identification, pole-placement by state feedback, decoupling, design of observers and model-matching.

Unlike the single variable case, the corresponding canonical forms for multivariable are not unique. This lack of uniqueness, beside tending to make their derivation more difficult, forces the design engineers to determine the best form among the several possibilities according to its practical advantages.

### 3.5.1 Controllable Canonical Form

If a given system  $[A, B, C]$  is controllable, then there exists a nonsingular matrix that can transform this system into a simple form called controllable canonical form.

#### 3.5.1.1 Single-Variable Case

Consider the  $n$ -dimensional linear time-invariant, single-variable dynamical equation:

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned}\tag{3.11}$$

where  $A$ ,  $b$  and  $c$  are, respectively,  $n \times n$ ,  $n \times 1$ , and  $1 \times n$ , and real constant matrices.

by introducing  $\bar{x} = Px$ , the following dynamical equation

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{b}u \\ y &= \bar{c}\bar{x}\end{aligned}\tag{3.12}$$

is an equivalent dynamical equation of (3.11). Then, we have

$$\bar{A} = PAP^{-1} \quad \bar{b} = Pb \quad \bar{c} = cP^{-1}$$

The controllability matrices of (3.11) and (3.12) are, respectively,

$$U_c = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}\tag{3.13}$$

and

$$\begin{aligned}\bar{U}_c &= [\bar{b} \quad \bar{A}\bar{b} \quad . \quad . \quad . \quad \bar{A}^{n-1}\bar{b}] \\ \bar{U}_c &= P[b \quad Ab \quad . \quad . \quad . \quad A^{n-1}b] = PU_c\end{aligned}\quad (3.14)$$

The characteristic polynomial of the matrix  $A$  in (3.11) is given

$$\Delta(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n \quad (3.15)$$

**Theorem 3.3:**

If the  $n$ -dimensional linear time-invariant, single-variable dynamical equation (3.11) is controllable, then it can be transformed, by an equivalence transformation, into the form

$$\begin{aligned}\dot{\bar{x}} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y &= [c_n \quad c_{n-1} \quad c_{n-2} \quad \dots \quad c_2 \quad c_1] \bar{x}\end{aligned}\quad (3.16)$$

where  $a_1, a_2, \dots, a_n$  are the coefficients of the characteristic polynomial of  $A$ , and  $c_i$ 's are to be computed from (3.11). The dynamical equation (3.16) is said to be in the controllable canonical form.

**Proof:** (see [6]).

### 3.5.1.2 Multivariable Case

Consider the  $n$ -dimensional linear time-invariant, multivariable dynamical equation

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\quad (3.17)$$

The determination of a constant nonsingular matrix  $P$  is required for the transformation  $\bar{x} = Px$  to produce a canonical form of the system with dynamical equation (3.17)

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x}\end{aligned}\tag{3.18}$$

The dynamical equations (3.17) and (3.18) are said to be equivalent and the matrix  $P$  is an equivalence transformation. In general, the system (3.17) is assumed to be completely controllable so that the standard condition (Barnett 1975) holds or equivalently the controllability matrix  $U_c$  has full rank.

Then to construct the matrix  $P$ , the matrix  $\Gamma_c$  defined in section 3.3 by (3.5) is inverted. Let  $\Gamma_c^{-1}$  be described by its rows as:

$$\Gamma_c^{-1} = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_n^T \end{bmatrix}\tag{3.19}$$

and let  $\gamma_{k_i}^T$  be the  $k_i$ th row of  $\Gamma_c^{-1}$ , where

$$k_i = \sum_{j=1}^i r_j \quad ; \quad i = 1, \dots, p\tag{3.20}$$

Using the vectors  $\gamma_{k_i}^T$ , the required transformation  $P$  is formed as:

$$P = \begin{bmatrix} \gamma_{k_1}^T \\ \gamma_{k_1}^T A \\ \vdots \\ \gamma_{k_1}^T A^{r_1-1} \\ \gamma_{k_2}^T \\ \vdots \\ \gamma_{k_p}^T A^{r_p-1} \end{bmatrix} \quad (3.21)$$

The controllable standard form for the system (3.17) is then given by:

$$\bar{A} = PAP^{-1} \quad (3.22)$$

$$\bar{B} = PB \quad (3.23)$$

$$\bar{C} = CP^{-1} \quad (3.24)$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \dots & \bar{A}_{1p} \\ \vdots & & \\ \bar{A}_{p1} & \dots & \bar{A}_{pp} \end{bmatrix} \quad (3.25)$$

with, for  $i = 1, \dots, p$ , the diagonal blocks  $\bar{A}_{ii}$  are of dimensions  $r_i \times r_i$  and are in companion form

$$\bar{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ \times & \times & \times & \dots & \times \end{bmatrix} \quad (3.26)$$

The off-diagonal blocks  $\bar{A}_{ij}$ ,  $i \neq j$ , are of dimensions  $r_i, r_j = 1, \dots, p, i \neq j$  have the form

$$\bar{A}_{ij} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 0 \\ \times & \times & \times & \dots & \times \end{bmatrix} \quad (3.27)$$

The matrix  $\bar{B}$  in (3.23) has the partitioned form

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \vdots \\ \bar{B}_p \end{bmatrix} \quad (3.28)$$

where

$$\bar{B}_i = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \times & \dots & \times \end{bmatrix} \quad (3.29)$$

↑  
column i

has dimension  $r_i \times p$ .

The  $\times$ 's in both  $\bar{A}$  and  $\bar{B}$  stand for possible nonzero entries.

The matrix  $\bar{C}$  has no special form under this transformation.

If  $\frac{n}{p} = l$  is an integer, and if the block controllability condition is satisfied, that is

$$U_c = [B \ AB \ A^2B \ \dots \ A^{l-1}B] \quad (3.30)$$

is full rank then a matrix  $P$  can be determined to transform the system to block controller form of index  $l$ . Where  $P$  is given by:

$$P = \begin{bmatrix} P_1 \\ P_1 A \\ P_1 A^2 \\ \vdots \\ P_1 A^{l-1} \end{bmatrix} \quad (3.31)$$

where

$$P_1 = \left[ \begin{array}{cccc} O_p & O_p & \dots & I_p \end{array} \right] \left[ \begin{array}{cccc} B & AB & \dots & A^{l-1}B \end{array} \right]^{-1} \quad (3.32)$$

The system in the new coordinates will be defined by the matrices of the form:

$$\bar{A} = \begin{bmatrix} O_p & I_p & \dots & O_p \\ O_p & O_p & \dots & O_p \\ \vdots & \vdots & & \\ O_p & O_p & \dots & I_p \\ A_r & A_{r-1} & \dots & A_1 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} O_p \\ O_p \\ \vdots \\ O_p \\ I_p \end{bmatrix} \quad (3.33)$$

where  $A_k$  are in controller companion form.

### 3.5.2 Observable Canonical Form

If a given system  $[A, B, C]$  is observable, then there exists a nonsingular matrix that can transform this system to a simple form called observable canonical form.

#### 3.5.2.1 Single-Variable Case

Consider the  $n$ -dimensional linear time-invariant, single-variable dynamical equation (3.11), by duality with controllability and introducing  $\bar{x} = Px$ , a dynamical equation of the form (3.12) is obtained.

The characteristic polynomial of the matrix  $A$  in (3.11) is given as

$$\Delta(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n \quad (3.34)$$

**Theorem 3.3:**

If the  $n$ -dimensional linear time-invariant, single-variable dynamical equation (3.11) is observable, then it can be transformed, by an equivalence transformation, into the form

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \bar{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \bar{x} \end{aligned} \quad (3.35)$$

where  $a_1, a_2, \dots, a_n$  are the coefficients of the characteristic polynomial of  $A$ , and  $b_i$ s are to be computed from (3.11). The dynamical equation (3.35) is said to be in the observable canonical form.

**Proof:** (see [6]).

### 3.5.2.2 Multivariable Case

Consider the  $n$ -dimensional linear time-invariant, multivariable dynamical equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (3.36)$$

The determination of a constant nonsingular matrix  $P$  is required for the transformation  $\bar{x} = Px$  to produce a canonical form of the system with dynamical equation (3.37).

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x}\end{aligned}\tag{3.37}$$

The dynamical equations (3.36) and (3.37) are said to be equivalent and the matrix  $P$  is an equivalence transformation. If the system (3.36) is completely observable then  $U_o$  has full rank.

To construct the transformation  $P$ , the matrix  $\Gamma_o$  defined in section 3.4 by (3.9) is inverted.

Let  $\Gamma_o^{-1}$  be described in term of its rows as:

$$\Gamma_o^{-1} = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_n^T \end{bmatrix}\tag{3.38}$$

and let  $\gamma_{k_i}^T$  be the  $k_i$ th row of  $\Gamma_o^{-1}$ , where

$$k_i = \sum_{j=1}^i r_j ; i = 1, \dots, q\tag{3.39}$$

Using the vectors  $\gamma_{k_i}^T$  the required transformation  $P$  is formed as:

$$P = \left[ \gamma_{k_1}, A\gamma_{k_1}, \dots, A^{r_1-1}\gamma_{k_1}, \gamma_{k_2}, A\gamma_{k_2}, \dots, A^{r_2-1}\gamma_{k_2}, \dots, \gamma_{k_q}, A\gamma_{k_q}, \dots, A^{r_q-1}\gamma_{k_q} \right]\tag{3.40}$$

The observable standard form for the system (3.36) is then given by:

$$\bar{A} = P^{-1}AP\tag{3.41}$$

$$\bar{B} = P^{-1}B\tag{3.42}$$

$$\bar{C} = CP\tag{3.43}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \dots & \bar{A}_{1q} \\ \vdots & & \\ \bar{A}_{q1} & \dots & \bar{A}_{qq} \end{bmatrix} \quad (3.44)$$

with, for  $i = 1, \dots, q$ , the diagonal blocks  $\bar{A}_{ii}$  are of dimensions  $r_i \times r_i$  and are in companion form

$$A_{ii} = \begin{bmatrix} 0 & 0 & \dots & 0 & \times \\ 1 & 0 & & 0 & \times \\ 0 & 1 & & 0 & \times \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & \times \end{bmatrix} \quad (3.45)$$

The off-diagonal blocks  $\bar{A}_{ij}$ ,  $i \neq j$ , are of dimensions  $r_i, r_j = 1, \dots, q, i \neq j$  have the form

$$\bar{A}_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 & \times \\ 0 & 0 & \dots & 0 & \times \\ \vdots & & & \vdots & \\ 0 & 0 & \dots & 0 & \times \\ 0 & 0 & \dots & 0 & \times \end{bmatrix} \quad (3.46)$$

The matrix  $\bar{C}$ , the transformed of the output matrix is:

$$\bar{C} = [\bar{C}_1 \quad \bar{C}_2 \quad \bar{C}_q] \quad (3.47)$$

where

$$\bar{C}_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \times \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \leftarrow \text{column } i \quad (3.48)$$

has dimension  $r_i \times q$

The  $\times$ 's in both  $\bar{A}$  and  $\bar{C}$  stand for possible nonzero entries.

The matrix  $\bar{B}$  has no special form under this transformation.

If  $\frac{n}{q} = m$  is an integer, and if the block controllability of index  $m$  condition is satisfied, that

is

$$U_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \dots & (A^T)^{m-1} C^T \end{bmatrix} \quad (3.49)$$

is full rank then a matrix  $P$  can be determined to transform the system to block observer form. Where  $P$  is given by:

$$P = \begin{bmatrix} P_1 & AP_1 & \dots & A^{m-1} P_1 \end{bmatrix} \quad (3.50)$$

where

$$P_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix}^{-1} \begin{bmatrix} O_q \\ O_q \\ \vdots \\ I_q \end{bmatrix} \quad (3.51)$$

The system in the new coordinates will be defined by the matrices of the form:

$$\bar{A} = \begin{bmatrix} O_q & O_q & \dots & O_q & A_m \\ I_q & O_q & \dots & O_q & A_{m-1} \\ O_q & I_q & & O_q & A_{m-2} \\ \vdots & \vdots & \ddots & & \vdots \\ O_q & O_q & & I_q & A_1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_m \\ B_{m-1} \\ B_{m-2} \\ \vdots \\ B_o \end{bmatrix} \quad (3.52)$$

$$\bar{C} = \begin{bmatrix} O_q & O_q & \dots & O_q & I_q \end{bmatrix}$$

where  $A_k$  are in observer companion form.

### 3.5.3 Jordan Canonical Form

Every linear time-invariant dynamical equation has an equivalent Jordan-form-dynamical equation.

Given a matrix  $A$ , for each eigenvalue of  $A$  with multiplicity  $m$ , there exist  $m$  linearly independent eigenvectors and generalized eigenvectors. Using these vectors as a basis, the new representation of  $A$  is in Jordan canonical form.

#### Theorem 3.4:

Each  $n \times n$  matrix  $A$  is similar to a Jordan form:  $J = P^{-1}AP$  and  $A = PJP^{-1}$ , with

$$P^{-1}AP = \begin{bmatrix} J_1 & O & \cdot & \cdot & \cdot & O \\ O & J_2 & \cdot & \cdot & \cdot & O \\ & & \cdot & \cdot & \cdot & \\ O & O & \cdot & \cdot & \cdot & J_\mu \end{bmatrix} = J \quad (3.53)$$

where each  $J_r$  is an  $n_r \times n_r$  Jordan block.

A Jordan block is a square upper-triangular matrix  $J(\lambda)$  such that:

1. All its main diagonal entries equal  $\lambda$ :  $\langle J(\lambda) \rangle_{ii} = \lambda$ .
2. All its entries on the first superdiagonal equal 1:  $\langle J(\lambda) \rangle_{ii+1} = 1$ .
3. All its other entries equal 0.

Thus

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda & 1 & \dots & \dots & 0 \\ & & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \dots & \dots & \lambda \end{bmatrix} \quad (3.54)$$

Apart from the ordering of the blocks along the diagonal of  $J$  (which can be arbitrary), the Jordan canonical form is unique.

**Proof:** (see [35])

### 3.6 Linear State Feedback

The problem of pole assignment has been the subject of many research papers over the last decades. It is question of moving the poles of a given time-invariant linear system to a specified set of locations using state feedback or output feedback methods.

Our work deals only with the state feedback design which can be used to control the eigenvalues of a dynamical equation. It considers the effect of a feedback law of the form  $u = r - Kx$  on a linear time-invariant dynamical equation, where  $r$  is the reference input and  $K$  is a real matrix.

The problem consists then of the determination of  $K$  such that the resulting closed-loop system

$$\dot{x} = (A - BK)x + Br \quad (3.55)$$

has a desired set of eigenvalues  $\{\lambda_i\}$ :  $i = 1, \dots, n$ .

According to Wonham [36], the condition for  $K$  to exist is that the system has to be completely controllable.

A large number of algorithms exist for the determination of  $K$ . In the remainder of this section, some of these methods will be reviewed for both single variable and multivariable cases.

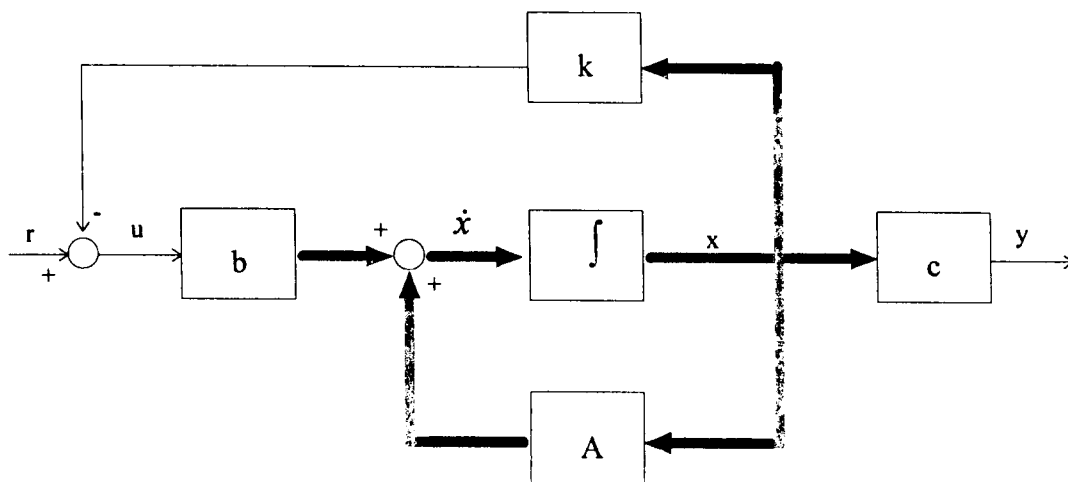
### 3.6.1 Single Input-Single Output

Consider the single-variable dynamical equation

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned}\tag{3.56}$$

State feedback consists of feeding back every state into the input terminal after being multiplied by a gain. If the gain between the  $i$ th state variable and the input terminal is defined as  $k_i$ , then  $k$  can be defined as  $k = [k_1 \ k_2 \ \dots \ k_n]$ . The dynamical equation of the state-feedback system is then given by the following equation:

$$\begin{aligned}\dot{x} &= (A - bk)x + br \\ y &= cx\end{aligned}\tag{3.57}$$



**Figure 3.2** A state feedback system

**Theorem 3.5:**

The state feedback dynamical equation (3.57) is controllable for any  $1 \times n$  real vector  $k$  if and only if the dynamical equation (3.56) is controllable.

**Proof:** (see [6]).

**Remark:**

The controllability property of a dynamical equation is preserved under state feedback but the observability can always be destroyed by some choice of  $k$ .

**3.6.1.1 Using the Transformation to Controller Canonical Form**

In this method, the dynamical equation (3.56) is transformed into controllable canonical form such that the new dynamical equation:

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{b}u \\ y &= c\bar{x}\end{aligned}\tag{3.58}$$

with  $\bar{A} = PAP^{-1}$  and  $\bar{b} = Pb$ , the state feedback is then given by  $u = r + kx = r + \bar{k}\bar{x}$  where  $\bar{k} = kP^{-1}$ . Since the original system and the new one are equivalent, then both have the same characteristic equation.

Let the desired polynomial be:

$$s^n + \bar{a}_1 s^{n-1} + \dots + \bar{a}_n$$

This desired characteristic polynomial is obtained by choosing  $\bar{k}$  as

$$\bar{k} = [a_n - \bar{a}_n \quad a_{n-1} - \bar{a}_{n-1} \quad \dots \quad a_1 - \bar{a}_1]\tag{3.59}$$

Therefore  $k$  is given by  $k = \bar{k}P$ , where  $\bar{k}$  is given by (3.59) and the determination of  $P$  has been introduced in section 3.5.1.

The above procedure is summarized in the following algorithm:

### Algorithm

Given a controllable  $(A, b)$  and a set of desired eigenvalues  $\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n$ . Find the  $1 \times n$  real vector  $k$  such that the closed loop system has the desired poles.

1. Find the characteristic polynomial of  $A$ :  $\det(sI - A) = s^n + a_1s^{n-1} + \dots + a_n$
2. Compute.  $(s - \bar{\lambda}_1)(s - \bar{\lambda}_2) \dots (s - \bar{\lambda}_n) = s^n + \bar{a}_1s^{n-1} + \dots + \bar{a}_n$ .
3. Compute  $\bar{k} = [a_n - \bar{a}_n \quad a_{n-1} - \bar{a}_{n-1} \quad \dots \quad a_1 - \bar{a}_1]$ .
4. Compute  $q_{n-i} = Aq_{n-i+1} + a_iq_n$ , for  $i=1, 2, \dots, (n-1)$ , with  $q_n = b$ .
5. Form  $Q = [q_1 \quad q_2 \quad \dots \quad q_n]$ .
6. Find  $P = Q^{-1}$ .
7.  $k = \bar{k}P$ .

### 3.6.1.2 Using Ackermann's Formula

Given a desired closed-loop characteristic polynomial

$$\Delta(s) = s^n + \bar{a}_1s^{n-1} + \dots + \bar{a}_n \quad (3.60)$$

Define  $\Delta(A)$  as the characteristic polynomial evaluated at  $A$ :

$$\Delta(A) = A^n + \bar{a}_1A^{n-1} + \dots + \bar{a}_nI$$

The feedback gain matrix is then given by:

$$k = [0 \quad 0 \quad \dots \quad 0 \quad 1]U^{-1}\Delta(A) \quad (3.61)$$

known under the name of Ackerman's formula and has been proved by Franklin and Powell [15] (1980),  $U$  being the controllability matrix.

### 3.6.2 Multiple Input-Multiple Output

Consider the multivariable uncompensated system described by the following dynamical equation:

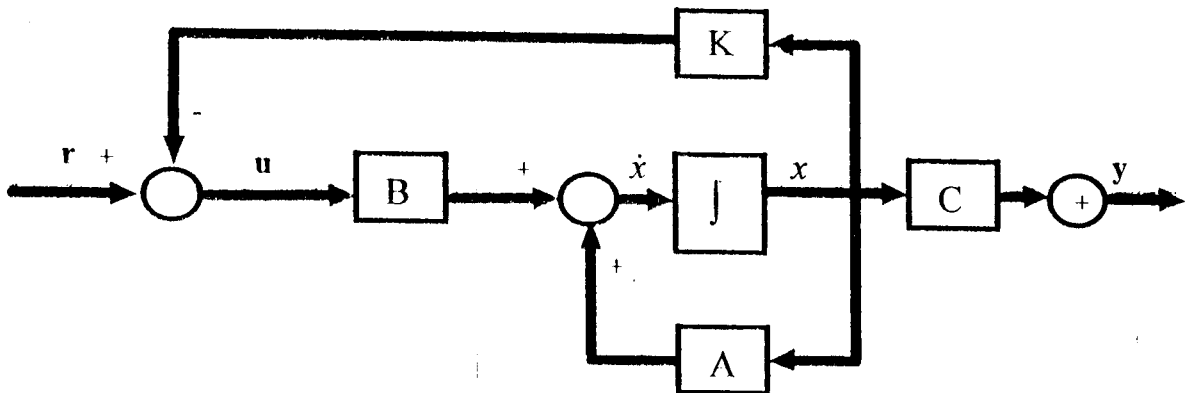
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\quad (3.62)$$

Where  $A$ ,  $B$  and  $C$  are respectively  $n \times n$ ,  $n \times p$  and  $q \times n$  constant real matrices.

If the input vector  $u$  in (3.62) is replaced by  $u = r - Kx$ , then the state equations of the compensated system are given by:

$$\begin{aligned}\dot{x} &= (A - BK)x + Br \\ y &= Cx\end{aligned}\quad (3.63)$$

A block diagram is shown below in order to illustrate the compensated system.



**Figure 3.3** A multivariable state feedback system

In the multivariable state feedback design, we are seeking a  $p \times n$  real matrix  $K$  such that the closed loop system has a desired set of poles (subject to complex pairing). In other words, the problem is to find  $K$  which gives the  $n$  desired coefficients of the desired characteristic polynomial of the compensated system.

There exist a large number of algorithms for the solution of the state feedback design problem. These algorithms can be divided into two categories according to the gain matrix which can be dyadic (has rank equal to one) or have full rank. Dyadic designs have been criticized due to the fact that closed-loop systems have poor disturbance rejection properties compared to the full-rank designs [36]. We will review in this section three methods which are the dyadic design, the full rank design and the design using the Lyapunov equation.

### 3.6.2.1 Dyadic Design

Before starting this section, let us first state two theorems which are essential in this context:

**Theorem 3.6:**

If the dynamical equation (3.62) is controllable, by a linear state feedback of the form  $u = r - Kx$ , where  $K$  is a  $p \times n$  real constant matrix, the eigenvalues of  $(A - BK)$  can be arbitrarily assigned provided that complex conjugate eigenvalues appear in pairs.

**Proof:** (see [6]).

**Theorem 3.7:**

If  $(A, B)$  is controllable and if  $A$  is cyclic, then for almost any  $p \times 1$  real vector  $f$  the single-input pair  $(A, Bf)$  is controllable.

**Proof:** (see [6]).

In the Dyadic design, two approaches can be followed: spectral or mapping. We'll see both in what follows.

#### *Spectral Approach*

The eigenvalues  $\{\lambda_i\}$ , the eigenvectors  $\{w_i\}$  and reciprocal eigenvectors  $\{v_i^t\}$  of the matrix  $A$  need to be calculated. The matrix  $K$  is taken as the product:

$$K = fm^t \quad (3.64)$$

where  $f$  is a  $p \times 1$  vector and  $m^t$  is a  $1 \times n$  vector.

The determination of the  $(p+n)$  free elements of  $K$  is nonlinear; to overcome this problem, values are preassigned to the elements of one vector resulting therefore in a linear problem to determine the remaining elements.

The vector  $f$  is chosen such that the pseudo-single-input system  $(A, Bf)$  is completely controllable; then for a desired set of eigenvalues  $\{\bar{\lambda}_i\}$ , the vector  $m$  is such that

$$m^t = \sum_{i=1}^g \delta_i v_i^t \quad (3.65)$$

where  $g$  is the number of poles to be altered,  $v_i^t$  are the reciprocal eigenvectors associated with eigenvalues  $\{\lambda_i\}$  to be altered, and  $\delta_i$  are weighting factors given by:

$$\delta_i = \frac{\prod_{j=1}^g (\lambda_i - \bar{\lambda}_j)}{p_i \prod_{\substack{j=1 \\ j \neq i}}^g (\lambda_i - \lambda_j)} \quad (3.66)$$

where the scalars  $p_i$  are the inner products

$$p_i = \langle v_i, Bf \rangle \quad (3.67)$$

### **Mapping Approach**

This approach, due to Young and Willems [33], unlike the spectral approach, does not require the spectral decomposition of the open-loop system.

Given a desired characteristic polynomial

$$\Delta_d(s) = s^n + d_1 s^{n-1} + \dots + d_n \quad (3.68)$$

and the open-loop system characteristic polynomial

$$\Delta(s) = s^n + a_1 s^{n-1} + \dots + a_n \quad (3.69)$$

we define a difference polynomial  $\delta(s)$  as

$$\begin{aligned} \delta(s) &= \Delta_d(s) - \Delta(s) \\ &= (d_1 - a_1)s^{n-1} + \dots + (d_n - a_n) \end{aligned} \quad (3.70)$$

A vector  $f$  is chosen such that  $(A, Bf)$  is completely controllable, and the required vector  $m^t$  is determined as

$$m = [\Phi_c^t]^{-1} X^{-1} \delta \quad (3.71)$$

with

$$\Phi_c = [Bf \quad ABf \quad \dots \quad A^{n-1}Bf] \quad (3.72)$$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & & \cdot \\ & a_1 & & \cdot \\ \vdots & \vdots & & \cdot \\ a_{n-2} & a_{n-3} & \dots & 1 \\ a_{n-1} & a_{n-2} & \dots & a_1 & 1 \end{bmatrix} \quad (3.73)$$

and

$$\delta = [d_1 - a_1 \quad d_2 - a_2 \quad \dots \quad d_n - a_n]^t \quad (3.74)$$

This latter algorithm can be approached from the frequency domain point of view. Given

$$\Gamma(s) = \text{adj}(sI - A) \quad (3.75)$$

and

$$b_f = Bf \quad (3.76)$$

with  $f$  again chosen such that  $(A, Bf)$  is completely controllable, we have

$$g(s) = \frac{1}{\Delta(s)} \Gamma(s) b_f = \frac{1}{\Delta(s)} \begin{bmatrix} N_1(s) \\ N_2(s) \\ \vdots \\ N_n(s) \end{bmatrix} \quad (3.77)$$

where

$$N_i(s) = \beta_{i1}s^{n-1} + \beta_{i2}s^{n-2} + \dots + \beta_{in} \quad (3.78)$$

and  $\Delta(s)$  has been defined by equation (3.69).

Let  $\Delta_d(s)$  defined as

$$\begin{aligned} \Delta_d(s) &= |sI - A + BK| \\ &= |sI - A + b_f m^t| \\ &= |sI - A| \left| I + (sI - A)^{-1} b_f m^t \right| \end{aligned} \quad (3.79)$$

which can be expressed as

$$\begin{aligned} \Delta_d(s) &= \Delta(s) + m^t \Gamma(s) b_f \\ &= \Delta(s) + \sum_{i=1}^n m_i N_i(s) \end{aligned} \quad (3.80)$$

From equation (3.70), the state feedback design problem can be considered as consisting of solving for the elements of the vector  $m$  in the following equation:

$$\sum_{i=1}^n m_i N_i(s) = \delta(s) \quad (3.81)$$

Solving by identification equation (3.81), we obtain

$$\begin{bmatrix} \beta_{11} & \cdot & \cdot & \cdot & \cdot & \beta_{n1} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \beta_{1n} & \cdot & \cdot & \cdot & \cdot & \beta_{nn} \end{bmatrix} \begin{bmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{bmatrix} \quad (3.82)$$

or

$$Xm = \delta \quad (3.83)$$

$m$  is then given by

$$m = X^{-1}\delta \quad (3.84)$$

which is consistent since  $X^{-1}$  always exists, because the rows of  $(sI - A)^{-1}b_f$  are linearly independent due to the fact that  $(A, b_f)$  is a controllable pair, hence,

$$(sI - A)^{-1}b_f = \frac{Adj(sI - A)}{\Delta(s)}b_f = \frac{1}{\Delta(s)}\Gamma(s)b_f,$$

and  $\delta$  is given by equation (3.74).

### 3.6.2.2 Full-Rank Design

It consists of transforming the pair  $(A, B)$  into controllable canonical form as described in section 3.5.1.2, and then removing the nonzero entries,  $\times$ , in the matrix  $\bar{B}$  by a simple

nonsingular transformation  $Q$  applied to the system inputs. This matrix is formed from the nonzero rows of the matrix  $PB$ .

$$\bar{A} = PAP^{-1} \quad (3.85)$$

$$\bar{B} = PBQ \quad (3.86)$$

Once the pair  $(\bar{A}, \bar{B})$  in multivariable controllable form is obtained, the  $K$  required to achieve a desired closed loop set of poles  $\{\lambda_i\}$  is chosen such that  $(\bar{A} - \bar{B}\bar{K})$  becomes block upper or lower triangular, or block diagonal and such that the resulting diagonal blocks have nonzero coefficients satisfying

$$\prod_{i=1}^p \det [sI - \bar{A}_{ii} + (\bar{B}\bar{K})_{ii}] = \prod_{i=1}^n (s - \bar{\lambda}_i) \quad (3.87)$$

The desired full-rank feedback matrix  $K$  in the original coordinate system is given by:

$$K = Q^{-1}\bar{K}P \quad (3.88)$$

### 3.6.2.3 Lyapunov Equation Method

The major step in the above method is the transformation of  $(A, B)$  into a multivariable controllable form. This requires equivalence transformations formed from the columns of the controllability matrix, in other words from the columns of  $A^k B$ ,  $k=1, 2, \dots, n-1$ . The application of such transformation may change the problem into a less well-conditioned problem. Therefore this method may not be satisfactory from a computational point of view.

The method we will see now does not require the transformation of an equation into controllable form, it is based on the solution of a Lyapunov type equation.

Consider a controllable  $(A, B)$  where  $A$  and  $B$  are respectively  $n \times n$  and  $n \times p$  constant real matrices. We need to find a  $K$  such that  $A - BK$  has a set of desired eigenvalues. The idea

is to set  $(A - BK)$  similar to a matrix  $F$  whose eigenvalues are the desired ones. This is equivalent to say that there exists a similarity transformation  $T$  such that  $A - BK = TFT^{-1}$  which implies  $AT - BKT = TF$  or  $AT - TF = B\bar{K}$  with  $\bar{K} = KT$ . The choice of the matrix  $F$  is arbitrary as long as it has no eigenvalue in common with those of  $A$ , this condition ensures the existence of a solution for the Lyapunov equation.

**Theorem 3.8:**

A necessary and sufficient condition for the equation  $AT - TF = B\bar{K}$  to have a solution for all  $G$  (with  $G = B\bar{K}$ ) is that  $\alpha_i - \lambda_i \neq 0$ , where  $\{\alpha_i\}$  are the characteristic roots of  $A$  and  $\{\lambda_i\}$  are the characteristic roots of  $F$ .

**Proof:** (see [3]).

A necessary condition for  $T$  to be nonsingular is that  $(A, B)$  is controllable and  $(F, \bar{K})$  is observable [6].

This method is illustrated in the following algorithm suggested by Chen [6].

**Algorithm**

Given a controllable pair  $(A, B)$ .

1. Choose an arbitrary  $n \times n$  matrix  $F$  which has no eigenvalues in common with those of  $A$ .
2. Choose an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  is observable.
3. Solve the unique  $T$  in the Lyapunov equation  $AT - TF = B\bar{K}$ .
4. If  $T$  is nonsingular, then we have  $K = \bar{K}T^{-1}$ , and  $A - BK$  has the same eigenvalues as those of  $F$ . If  $T$  is singular, choose a different  $F$  or a different  $\bar{K}$  and repeat the process.

**4.1 Introduction**

In the present chapter we will have a look at the solution of the Lyapunov (Sylvester) equation; it is an important matrix equation which is often encountered in many fields beside control system theory. Many papers have been published for this purpose; we will present here some approaches found in Linear Algebra and the approach we will use to solve our problem. The general form of the so-called Lyapunov (Sylvester) equation is  $AX + XB = C$ .

**4.2 Solution given by integration**

Consider the equation:

$$AX + XB = C \quad (4.1)$$

If all eigenvalues of the matrix  $A \in C^{m \times m}$  and  $B \in C^{n \times n}$  have negative real parts, then the unique solution  $X$  of the Lyapunov equation is given by [28].

$$X = - \int_0^{\infty} e^{At} C e^{Bt} dt$$

**4.2.1 The problem of calculating  $e^{At}$  ( $e^{Bt}$ )**

- $e^{At}$  can be calculated directly from the series definition  $e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ . Although this

matrix series is convergent for all  $A$ , its computation remains a hopeless task for most cases (unless  $A$  is a matrix whose powers may be readily calculated).

- Transform  $A$  into Jordan canonical form then  $e^{At}$  can be written as

$$e^{At} = P e^{Jt} P^{-1} = P \begin{bmatrix} e^{J_1 t} & & & & \\ & e^{J_2 t} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & e^{J_\sigma t} \end{bmatrix} P^{-1}$$

- Another method consists in using the Laplace Transform as seen from the fact that  $L[e^{At}] = [sI - A]^{-1}$ . Thus, to obtain  $e^{At}$ : first invert  $[sI - A]$ , this results in a matrix whose elements are rational functions of  $s$ , then take the inverse Laplace transform of each element of the matrix.

**Remark:** The fact that this method requires the computation of  $[sI - A]^{-1}$  which is computationally costly, makes it impractical.

#### 4.2.2 Numerical Integration

Once  $e^{At}$  and  $e^{Bt}$  have been calculated, the solution of the Lyapunov equation will be given by:

$$X = - \int_0^{\infty} D(t) dt$$

with  $D(t) = e^{At} C e^{Bt}$  whose elements are functions of  $t$ . To compute the solution then, the elements of  $D(t)$  need to be integrated; for this purpose, one of the numerical integration methods must be used to obtain an approximate value.

We can conclude that the solution by integration cumulates the computational cost of evaluating  $e^{At}$  and of numerical integration which makes it costly. Also, this method relies on the stability of the matrices  $A$  and  $B$  which is generally not ensured.

### 4.3 Solution Via the Kronecker Product

In this section we will discuss some results concerning the Kronecker product also known as direct product or tensor product.

#### 4.3.1 Definition of the Kronecker Product

It is a binary operation from  $C^{m \times l} \times C^{n \times k}$  to  $C^{mn \times lk}$ . In the case where  $l=m$ ,  $k=n$

$A = [a_{ij}]_{i,j=1}^m \in C^{m \times m}$ ,  $B = [b_{ij}]_{i,j=1}^n \in C^{n \times n}$  then the right Kronecker product of  $A$  and  $B$ ,  $A \otimes B$  is defined to be

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdot & \cdot & \cdot & a_{1m}B \\ a_{21}B & a_{22}B & \cdot & \cdot & \cdot & a_{2m}B \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{m1}B & a_{m2}B & \cdot & \cdot & \cdot & a_{mm}B \end{bmatrix} = [a_{ij}B]_{i,j=1}^m \in C^{mn \times mn}$$

The proofs of the following propositions and corollaries can be found in [28].

**Proposition 1**

a) If  $\mu \in \mathbb{C}$   $(\mu A) \otimes B = A \otimes (\mu B) = \mu(A \otimes B)$

b)  $(A + B) \otimes C + (A \otimes C) + (B \otimes C)$

c)  $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

d)  $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

e)  $(A \otimes B)^T = A^T \otimes B^T$

**Proposition 2**

If  $A, C \in \mathbb{C}^{m \times m}$  and  $B, D \in \mathbb{C}^{n \times n}$ , then

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

**Corollary 1**

If  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{n \times n}$ , then

a)  $A \otimes B = (A \otimes I_n)(I_m \otimes B) = (I_m \otimes B)(A \otimes I_n)$

b)  $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$  provided that  $A^{-1}$  and  $B^{-1}$  exist.

**Corollary 2**

If  $A_1, A_2, \dots, A_p \in \mathbb{C}^{m \times m}$  and  $B_1, B_2, \dots, B_p \in \mathbb{C}^{n \times n}$ , then

$$(A_1 \otimes B_1)(A_2 \otimes B_2) \dots (A_p \otimes B_p) = (A_1 A_2 \dots A_p) \otimes (B_1 B_2 \dots B_p)$$

The Kronecker product is, in general, not commutative.

**Proposition 3**

If  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{n \times n}$ , then there exists a permutation matrix  $P \in \mathbb{R}^{mn \times mn}$  such that

$$P^T(A \otimes B)P = B \otimes A$$

#### Proposition 4

If  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{n \times n}$ ,  $X \in \mathbb{C}^{m \times n}$ , then

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}X$$

#### Corollary 1

With the above notation,

$$\text{a) } \text{vec}(AX) = (I_n \otimes A)\text{vec}X;$$

$$\text{b) } \text{vec}(XB) = (B^T \otimes I_n)\text{vec}X;$$

$$\text{c) } \text{vec}(AX + XB) = \left( (I_n \otimes A) + (B^T \otimes I_m) \right) \text{vec}X.$$

### 4.3.2 Application of the Kronecker Product to Matrix Equations:

Consider the general linear matrix equation

$$A_1XB_1 + A_2XB_2 + \dots + A_pXB_p = C, \quad (4.2)$$

where  $A_j \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{m \times m}$  ( $j = 1, 2, \dots, p$ ),  $X, C \in \mathbb{C}^{m \times m}$ .

The solution of this equation relies on proposition 4 which permits the reduction of the original equation to a matrix-vector equation of the form  $Gx = c$  where  $G \in \mathbb{C}^{mn \times mn}$  and  $x, c \in \mathbb{C}^{mn}$ .

#### Theorem 4.1:

A matrix  $X \in \mathbb{C}^{m \times n}$  is a solution of equation (4.2) if and only if the vector  $x = \text{vec}X$  is the solution of equation

$$Gx = c \quad (4.3)$$

with  $G = \sum_{j=1}^p (B_j^T \otimes A_j)$  and  $c = \text{vec}C$

**Proof:** (see [28]).

**Corollary 1:**

Equation (4.2) has a solution  $X$  if and only if  $\text{rank}[G \ c] = \text{rank} \ G$

Equation (4.2) has a unique solution if and only if the matrix  $G$  in equation (4.3) is nonsingular.

A particular case of equation (4.2) is the matrix equation

$$AX + XB = C \tag{4.4}$$

**Proof:** (see [28]).

**Theorem 4.2:**

Equation (4.4) has a unique solution if and only if the matrices  $A$  and  $-B$  have no eigenvalues in common.

**Proof:** (see [20, 28]).

The transformation of the Lyapunov equation using the Kronecker product results in a set of linear equations of the form  $Gx = c$ . Although a straightforward solution can be obtained using one of the existing methods for solving such a matrix equation, this technique remains not recommended considering the cost of transforming the system into a linear equation, and the dimensions of the resulting system of equations.

#### 4.4 Bartels-Stewart Algorithm

Consider the equation (4.1), the crux of the Bartels-Stewart algorithm is to use the QR algorithm to compute the real Schur decompositions

$$\begin{aligned} U^T A U &= R \\ V^T B^T V &= S \end{aligned} \quad (4.5)$$

where  $R$  and  $S$  are upper quasi-triangular and  $U$  and  $V$  are orthogonal.

The reductions (4.5) lead to the transformed system

$$R Y + Y S^T = F \quad \left( F = U^T C V, \quad Y = U^T X V \right) \quad (4.6)$$

Assuming  $s_{k,k-1}$  is zero, then it follows that

$$(R + s_{kk} I) y_k = f_k - \sum_{j=k+1}^n s_{kj} y_j \quad (4.7)$$

where  $Y = [y_1 \mid y_2 \mid \cdots \mid y_n]$  and  $F = [f_1 \mid f_2 \mid \cdots \mid f_n]$ . Thus  $y_k$  can be found from  $y_{k+1}, \dots, y_n$  by solving an upper quasi-triangular system. If  $s_{k,k-1}$  is nonzero, then  $y_k$  and  $y_{k-1}$  are simultaneously found in an analogous fashion.

#### 4.5 Hessenberg-Schur Algorithm

This method is based upon the equivalence of the problem  $AX + XB = C$  and  $(U^{-1} A U)(U^{-1} X V) + (U^{-1} X V)(V^{-1} B V) = U^{-1} C V$ .

The matrices  $A$  and  $F$  are decomposed as follows:

$$\begin{aligned} H &= U^T A U & H \text{ upper Hessemberg,} & & U \text{ orthogonal} \\ S &= V^T B^T V & S \text{ quasi-upper triangular,} & & V \text{ orthogonal} \end{aligned} \quad (4.8)$$

The reduction (4.8) leads to a system of the form

$$H Y + Y S^T = F \quad \left( F = U^T C V \right) \quad (4.9)$$

If  $s_{k,k-1} = 0$  then  $y_k$  is determined by solving the  $n \times n$  Hessenberg system:

$$(H + s_{k,k}I)y_k = f_k - \sum_{j=k+1}^n s_{k,j}y_j$$

If  $s_{k,k-1}$  is nonzero, then by equating columns  $k-1$  and  $k$  in (4.9), we find

$$H[y_{k-1}|y_k] + [y_{k-1}|y_k] \begin{bmatrix} s_{k-1,k-1} & s_{k,k-1} \\ s_{k-1,k} & s_{k,k} \end{bmatrix} = [f_{k-1}|f_k] \\ - \sum_{j=k+1}^n [s_{k-1,j}y_j|s_{k,j}y_j]$$

Summarising the algorithm, we have the following:

1. Reduce  $A$  to upper Hessenberg and  $B^T$  to quasi-upper triangular.

$$H = U^T A U$$

$$S = V^T B^T V$$

2. Update the right-hand side:

$$F = U^T C V$$

3. Back substitution for  $Y$ :

$$H Y + Y S^T = F$$

4. Obtain solution:

$$X = U Y V^T$$

## 5.1 Introduction

The contribution of this thesis consists of the proposition of a complete algorithm for the multivariable state feedback design. It consists of Chen's algorithm with an elaboration of a method for solving the Lyapunov equation where the most important point is the choice of the arbitrary matrices; it will be shown that these matrices could be chosen under certain forms that will ensure the nonsingularity of the solution of the Lyapunov equation whose inverse will be used in the determination of the feedback matrix. Thus reducing the cost of the whole algorithm.

## 5.2 The Proposed Algorithm

Consider a controllable pair  $(A, B)$  where  $A$  and  $B$  are, respectively,  $n \times n$  and  $n \times p$  constant matrices, find a  $K$  such that  $(A - BK)$  has a set of desired eigenvalues.

Choose an arbitrary pair of matrices  $(F, \bar{K})$ . The choice of the matrices  $F$  and  $\bar{K}$  is almost arbitrary which gives a wide range of structures. The only constraints imposed on the choice of  $F$  and  $\bar{K}$  are:

- $A$  and  $F$  must have no common eigenvalues to ensure a solution to the equation.
- $(F, \bar{K})$  must be observable, which is a necessary condition for the nonsingularity of the solution  $T$ .

The goal is two-fold: propose particular forms for  $F$  and  $\bar{K}$  to:

- converge at first try and
- to reduce the amount of computations required by the Hessenberg Schur algorithm.

**The Algorithm:**

1. Choose an arbitrary  $n \times n$  matrix  $F$  which has no eigenvalues in common with those of  $A$ .
2. Choose an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  observable.
3. Reduce  $A$  to upper Hessenberg and  $F^T$  to quasi-upper triangular

$$\begin{aligned} H &= U^T A U \\ S &= V^T F^T V \end{aligned} \tag{5.1}$$

4. Update the right-hand side

$$G = U^T B \bar{K} V \tag{5.2}$$

5. Back substitution for  $Y$ :

$$HY - YS^T = G \tag{5.3}$$

6. Obtain the solution:

$$T = UYV^T \tag{5.4}$$

7. Obtain the feedback matrix:

$$K = \bar{K} T^{-1} \tag{5.5}$$

Given the desired poles, the matrix  $F$  will be constructed under one of the forms suggested in this thesis, Jordan form or controller form. The following section will deal with the observability of the couple formed by the arbitrary matrices  $(F, \bar{K})$ .

### 5.3 Observability of $(F, \bar{K})$

This section will deal with the conditions under which the pair  $(F, \bar{K})$  is observable for both Jordan and Controller canonical forms.

#### 5.3.1 Observability of Jordan Canonical Form

If  $F$  is chosen to be in Jordan form then what form should  $\bar{K}$  have in order to ensure that the pair  $(F, \bar{K})$  is observable? According to [6], if  $F$  is in Jordan form, the observability of the couple  $(F, \bar{K})$  can be checked by inspection.

Let the  $n \times n$  matrix  $F$  be in Jordan form, with  $m$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ .  $F_i$  denotes all the Jordan blocks associated with the eigenvalue  $\lambda_i$ ,  $r(i)$  is the number of Jordan blocks in  $F_i$  and  $F_{ij}$  is the  $j^{\text{th}}$  Jordan block in  $F_i$ . Clearly

$$F_i = \text{diag}(F_{i1}, F_{i2}, \dots, F_{ir(i)}) \text{ and}$$

$$F = \text{diag}(F_1, F_2, \dots, F_m)$$

$$F_{(n \times n)} = \begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_m \end{bmatrix}$$

$$\bar{K} = [K_1 \quad K_2 \quad \dots \quad K_m]$$

$$F_i = \begin{bmatrix} F_{i1} & & & \\ & F_{i2} & & \\ & & \ddots & \\ & & & F_{ir(i)} \end{bmatrix}$$

$$K = \begin{bmatrix} K_{i1} & K_{i2} & \cdots & K_{ir(i)} \end{bmatrix}$$

$$F_{ij} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & & \ddots & \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{bmatrix}$$

$$K_{ij} = \begin{bmatrix} K_{1ij} & K_{2ij} & \cdots & K_{lij} \end{bmatrix}$$

Let  $n_i$  and  $n_{ij}$  be the order of  $F_i$  and  $F_{ij}$  respectively; then

$$n = \sum_{i=1}^m n_i = \sum_{i=1}^m \sum_{j=1}^{r(i)} n_{ij}$$

corresponding to  $F_i$  and  $F_{ij}$ , the matrix  $\bar{K}$  is partitioned as shown. The first column of  $K_{ij}$  is denoted by  $k_{lij}$ .

**Theorem 5.1:**

$(F, \bar{K})$  is observable if and only if for each  $i=1,2,\dots,m$ , the columns of the  $q \times r(i)$  matrix

$$K_i^1 = \begin{bmatrix} K_{li1} & K_{li2} & \cdots & K_{lir(i)} \end{bmatrix}$$

are linearly independent.

**Proof:** (see [6]).

**Illustration:**

Consider

$$F = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & \lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & & \lambda_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & \lambda_{m-1} & & & \\ & & & & & & \lambda_m & 1 & 0 \\ & & & & & & & \lambda_m & 1 \\ & & & & & & & & \lambda_m \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 1 & 0 & 0 & 2 & & 1 & 0 \\ 0 & 0 & 1 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ k_{111} & k_{112} & k_{121} & & k_{1(m-1)1} & k_{1m1} \end{array}$$

The matrix  $F$  has  $m$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . There are two Jordan blocks associated with  $\lambda_1$ ; hence  $r(1)=2$ . There is only one Jordan block associated with  $\lambda_m$ ; hence  $r(m)=1$ .

The condition for  $(F, \bar{K})$  to be observable is that the sets  $\{k_{111}, k_{112}\}; \dots; \{k_{1m1}\}$  be individually linearly independent. Since the set  $\{k_{1m1}\}$  consists of a zero vector, it is linearly dependent. Hence  $(F, \bar{K})$  is not observable.

**Complex eigenvalues:**

The Jordan form for a matrix with complex eigenvalues in conjugate pairs has a slightly different form compared to the case of real eigenvalues, and the observability condition is still valid. For clarity consider the following 8x8 matrix  $F$ :

$$F = \begin{bmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & -\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$                      $\uparrow$                      $\uparrow$                      $\uparrow$   
 $k_{111}$                  $k_{112}$                  $k_{121}$                  $k_{131}$

There are four Jordan blocks: two corresponding to the pair  $(\alpha_1 \pm \beta_1)$ , one corresponding to  $(\alpha_2 \pm \beta_2)$  and one corresponding to  $\lambda_3$ .

For the above scheme, the observability is satisfied if and only if the sets  $\{k_{111}, k_{112}\}$ ;  $\{k_{121}\}$  and  $\{k_{131}\}$  are individually linearly independent. The observability is satisfied for this scheme since the linearity independence is verified.

**Example:**

Consider a matrix  $F$  with eigenvalues  $(-1 \pm 2i)$ ;  $(-1 \pm 2i)$  and  $-2$  such that

$$F = \begin{bmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$              $\uparrow$              $\uparrow$   
 $k_{111}$          $k_{112}$          $k_{121}$

Rank of  $\{k_{111}, k_{112}\}$  is two and  $\{k_{121}\}$  is linearly independent, thus  $(F, \bar{K})$  is observable.

**5.3.2 Observability of Controller Canonical Form**

In general, Chen's transformation gives the following form of  $(F, \bar{K})$

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1\mu} \\ F_{21} & F_{22} & \cdots & F_{2\mu} \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & \\ F_{\mu 1} & F_{\mu 2} & \cdots & F_{\mu\mu} \end{bmatrix} \quad (5.7)$$

$$\bar{K} = [K_1 \quad K_2 \quad \cdots \quad K_\mu]$$

for  $i=j$ ,

$$F_{ij} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ x & x & x & \cdots & x \end{bmatrix}$$

for  $i \neq j$

$$F_{ij} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \\ x & x & x & & x \end{bmatrix}$$

For convenience if for  $i \neq j$ , the arbitrary blocks  $F_{ij}$  are chosen as null blocks, the matrix  $F$  will have a block diagonal form. Thus, (5.7) will have the form below:

$$F = \begin{bmatrix} F_{11} & & \cdots & \\ & F_{22} & & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ & & & \cdots \\ & & & \cdots & F_{\mu\mu} \end{bmatrix} \quad (5.8)$$

$$\bar{K} = \begin{bmatrix} K_1 & K_2 & \cdots & K_\mu \end{bmatrix}$$

In order to check the observability of  $(F, \bar{K})$ , let us consider this system as a system formed by parallel connection of  $\mu$  subsystems.

A necessary condition for the resulting system to be observable, is that the constituent systems  $\{(F_1, K_1), (F_2, K_2), \dots, (F_\mu, K_\mu)\}$  should be so individually. This condition is sufficient if the sets of eigenvalues associated with the subsystems are disjoint [14]. Otherwise, this condition may not be sufficient and in this case, the first columns of the blocks in  $K$  corresponding to the blocks of  $F$  that have common eigenvalues are required to be linearly independent.

For the observability of each system individually, it is clear that when a system is in controller form, it is enough to have one element of the output matrix to extract the states. Thus, in general  $(F_k, K_k)$ , with  $F_k$  in controller form, is observable if  $K_k$  is non null.

**Example:**

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 3 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix}$$

For the pair  $(F, \bar{K})$  to be observable, it is required that the three blocks of  $\bar{K}$  be all three nonzero blocks.

$$\bar{K} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If, for instance the two (2x2) blocks are taken having the same eigenvalues, resulting in an  $F$  such that

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 3 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

then, in order for our pair to be observable, it is required that the first block of  $\bar{K}$  to be nonzero, and the first columns of the two last blocks to be linearly independent. Thus, for

$$\bar{K} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the observability of  $(F, \bar{K})$  is satisfied.

#### 5.4 Jordan form approach

Given a set of desired eigenvalues  $\{\lambda_i\}$ , construct the matrix  $F$  in Jordan form and an appropriate matrix  $\bar{K}$  so that to satisfy the observability condition.

The following equation is then obtained:

$$AT - TF = B\bar{K} \quad (5.9)$$

The next step is then to transform  $A$  into Hessenberg form  $H$  such that

$$H = U^T AU$$

$F$  normally needs to be transformed into a matrix in Schur form,  $S$  such that

$$S = V^T F^T V$$

But, since  $F$  is in Jordan form, it makes it already in Schur form; therefore, there is no need to transform it, while  $V$  is the identity matrix. Thus,

$$S = IF^T I$$

Equation (5.9) becomes then

$$UHU^T T - TS^T = B\bar{K} \quad (5.10)$$

Instead of premultiplying by  $U^T$  and postmultiplying by  $V$ , we need only the premultiplication by  $U^T$ . Setting  $Y = U^T T$  and  $G = U^T B\bar{K}$  equation (5.10) becomes

$$HY - YS^T = G \quad (5.11)$$

The elements of  $S$  are such that for the real eigenvalues,  $s_{k,k-1} = 0$ , the intermediate solution  $Y$  will be deduced by solving the  $p \times p$  Hessenberg system

$$(H + s_{kk}I)y_k = g_k - \sum_{j=k+1}^n s_{kj}y_j \quad (5.12)$$

And for the complex eigenvalues the corresponding columns of  $Y$  will be determined by solving the  $2p \times 2p$  linear system for  $y_k$  and  $y_{k-1}$

$$\begin{aligned} H[y_{k-1}|y_k] + [y_{k-1}|y_k] \begin{bmatrix} s_{k-1,k-1} & s_{k,k-1} \\ s_{k-1,k} & s_{k,k} \end{bmatrix} &= [g_{k-1}|g_k] \\ &- \sum_{j=k+1}^n [s_{k-1,j}y_j|s_{kj}y_j] \end{aligned} \quad (5.13)$$

Finally the solution is obtained by:

$$T = UY \quad (5.14)$$

Our goal we said before, is to reduce the cost of the algorithm and also to ensure that a solution is obtained at the first execution of the algorithm; this is what will be discussed next. It is understood that the problem is equivalent to set the new system's matrix  $(A - BK)$  similar to a matrix  $F$  whose eigenvalues are the desired poles, this is equivalent to say that there exists a similarity transformation that relates  $(A - BK)$  and  $F$  such that:

$$T^{-1}(A - BK)T = F$$

Every  $n \times n$  matrix is similar to a matrix in Jordan canonical form. Taking  $F$  in Jordan form means that  $T$  will be formed by the left and right eigenvectors of the feedback system  $(A - BK)$ . This ensures the existence of a nonsingular (modal) matrix although far from being unique. The uniqueness of  $T$  is ensured in our case by  $A$  and  $F$  not having common eigenvalues.

Therefore, given  $F$  in Jordan form permits to skip its transformation into a Schur form  $S$  as well as the multiplication by  $V$  since it is an identity matrix, so the work count is considerably reduced. On the other hand, as mentioned earlier, this form ensures a nonsingular solution for the Lyapunov equation which is a very important fact.

## 5.5 Controller form approach

### 5.5.1 $\frac{n}{p}$ is not an integer

Given a set of desired eigenvalues  $\{\lambda_i\}$ , the characteristic equation is first determined. After determining the number of blocks to be used, construct  $F$  in general controller form and construct an appropriate matrix  $\bar{K}$  so that they form an observable couple. The following equation will then have to be solved

$$AT - TF = B\bar{K} \tag{5.15}$$

Transform  $A$  into Hessenberg form  $H$  such that

$$H = U^T A U$$

Transform  $F^T$  into Schur form  $S$ . The Schur form is obtained by first reducing the original matrix to Hessenberg form then obtaining the Schur form. Since  $F^T$  is already in Hessenberg form, the first operation will be skipped.

$$S = V^T F^T V$$

Equation (5.15) becomes then

$$HY - YS^T = G \quad (5.16)$$

The columns of the intermediate solution  $Y$  will be determined by solving the  $p \times p$  Hessenberg system

$$(H + s_{kk}I)y_k = g_k - \sum_{j=k+1}^n s_{kj}y_j$$

if  $s_{k,k-1} = 0$ , and if  $s_{k,k-1} \neq 0$ , by solving the  $2p \times 2p$  linear system for  $y_k$  and  $y_{k-1}$

$$H \begin{bmatrix} y_{k-1} \\ y_k \end{bmatrix} + \begin{bmatrix} y_{k-1} \\ y_k \end{bmatrix} \begin{bmatrix} s_{k-1,k-1} & s_{k,k-1} \\ s_{k-1,k} & s_{k,k} \end{bmatrix} = \begin{bmatrix} g_{k-1} \\ g_k \end{bmatrix} - \sum_{j=k+1}^n \begin{bmatrix} s_{k-1,j}y_j \\ s_{k,j}y_j \end{bmatrix}$$

Finally the solution is obtained as:

$$T = UYV^T$$

We can conclude that the controller form has the advantage of being already in Hessenberg form which enables to reduce the effort in the transformation to Schur form.

Concerning the nonsingularity of the solution, bearing in mind that a necessary condition for feedback design is that the uncompensated system has to be controllable, and, that the controllability property of a dynamical equation is preserved by state-feedback,  $(A - BK)$  can always be transformed to controller form, in other words, there always exists a linearity transformation  $T$  such that

$$T^{-1}(A - BK)T = F$$

where  $F$  is in controller form and  $T$  of course is nonsingular. This ensures that the algorithm has to be executed only once to get a feedback matrix  $K$ .

### 5.5.2 $\frac{n}{p}$ is an integer

In case  $\frac{n}{p}$  is an integer, Chen's transformation gives the following form of  $(F, \bar{K})$

$$F = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -F_\mu & -F_{\mu-1} & -F_{\mu-2} & \cdots & -F_1 \end{bmatrix}$$

$$\bar{K} = [K_\mu \quad K_{\mu-1} \quad K_{\mu-2} \quad \cdots \quad K_1]$$

with the following characteristic equation

$$I\lambda^\mu + F_1\lambda^{\mu-1} + F_2\lambda^{\mu-2} + \dots + F_\mu \quad (5.17)$$

The blocks  $F_k$  are determined by equating the above characteristic equation with the matrix polynomial below:

$$(\lambda I - R_1)(\lambda I - R_2) \dots (\lambda I - R_\mu) \quad (5.18)$$

where the  $R_i$ 's are the desired spectral factors with eigenvalues being the desired eigenvalues or eigenvalues of  $F$ .

This matrix does not have a particular form suitable for the Hessenberg Schur algorithm. The required Schur decomposition will in general destroy the sparsity of the matrix [22]. For this reason, we are not interested in using this form in this work.

## 5.6 The Proposed Block Form Algorithm

For both the block Jordan form and the general block controller form,  $F$  has the following form:

$$F = \begin{bmatrix} F_1 & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & F_2 & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & F_\mu \end{bmatrix}$$

where for the Jordan form case

$$F_k = \begin{bmatrix} \lambda_k & 1 & & & \\ & \lambda_k & 1 & & \\ & & & \ddots & \\ & & & & \lambda_k & 1 \\ & & & & & \lambda_k \end{bmatrix}$$

and for the controller form case

$$F_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ x & x & x & \dots & x \end{bmatrix}$$

If the matrix  $F$  is dealt with as block diagonal, developing the Lyapunov equation will give a system of  $\mu$  Lyapunov equations such that:

$$\begin{cases} AT_1 - T_1 F_1 = B\bar{K}_1 \\ AT_2 - T_2 F_2 = B\bar{K}_2 \\ \vdots \\ AT_\mu - T_\mu F_\mu = B\bar{K}_\mu \end{cases}$$

The Hessenberg-Schur algorithm will be used to solve each equation of the above system determining this way the solution  $T$  column block by column block, respectively. The following algorithm is given to illustrate the steps of this procedure.

**The algorithm:**

Reduce  $A$  to upper Hessenberg;

$$H = U^T A U \quad (5.19)$$

If  $F$  is written in block form discussed above, the Lyapunov equation will then be decomposed into  $\mu$  equations having the same form;

$$AT_k - T_k F_k = B\bar{K}_k \quad (5.20)$$

to be solved for  $T_k$ ,  $k = 1, 2, \dots, \mu$ .

Reduce  $F_k$  to Schur form;

$$F_k = V_k S_k^T V_k^T \quad (5.21)$$

Update the right hand side of (5.20);

$$G_k = U^T B\bar{K}_k V_k \quad (5.22)$$

Solve for  $Y_k$  the equations:

$$HY_k - Y_k S_k^T = G_k \quad (5.23)$$

Obtain the column blocks of the solution

$$T_k = UY_kV_k^T \quad (5.24)$$

Compute the feedback matrix by solving for  $K$  the equation below;

$$KT = \bar{K} \quad (5.23)$$

Using this algorithm, we will see later, in the next chapter the effect of the number of blocks on the number of operations and the used space.

## 6.1 Introduction

In this chapter, we will test the algorithms described previously. For this purpose, we have developed a set of computer programs using MATLAB, the well known software for handling matrices. These algorithms will be evaluated according to their stability and cost.

## 6.2 Algorithms

This section will deal with the description and comparison of the results obtained using the proposed algorithms. They consist of the original Chen's algorithm to which we have incorporated the Hessenberg-Schur method to solve the resulting Lyapunov equation, and this for general matrices  $F$  and  $\bar{K}$ , and, for the proposed forms. Finally, we will present an extension of Chen's algorithm to block matrices, the "block Chen's algorithm".

### 6.2.1 The General Algorithm

It consists of choosing an arbitrary  $n \times n$  matrix  $F$  (having the desired eigenvalues) and an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  is observable, then reducing  $A$  and  $F$  respectively to Hessenberg and Schur forms,  $H = U^T A U$  and  $S = V^T F^T V$ . We form the matrix  $G = U^T B \bar{K} V$  and solve the system  $HY - YS^T = G$  for the intermediate solution  $Y$ . The solution  $T$  is finally obtained as  $T = U Y V^T$ .

If this solution is nonsingular then the feedback matrix  $K$  is obtained by solving  $KT = \bar{K}$ , otherwise the process is repeated for a different  $F$  and/or a different  $\bar{K}$ .

**The cost:** The transformations of  $A$  and  $F^T$  require respectively about  $\left(\frac{5}{3}n^3\right)$  and  $(10n^3)$  operations. To form the matrix  $G$ ,  $(3n^2p)$  flops are needed (multiplication of four matrices of dimension  $n \times n$ ,  $n \times p$ ,  $p \times n$  and  $n \times n$  in this order). For the next step, which consists of determining  $Y$ , the operations count depends actually on the form of  $S$ : if it is assumed to have  $\left(\frac{n}{2}\right)(2 \times 2)$  bumps along the diagonal, which is the worst case, and if Gaussian elimination with partial pivoting is used, then an order of  $\left(\frac{7}{2}n^3\right)$  operations are required. The solution  $T$  is obtained in  $(2n^3)$  flops. In order to determine whether or not  $T$  is singular, the singular value decomposition can be used, this process requires the order of  $(n^3)$  flops [27]. Hence, the total operations count at this stage is of the order of  $\left(\frac{109}{6}n^3 + 3n^2p\right)$  flops.

If  $T$  is singular, the above operations are repeated for a different  $F$  and/or a different  $\bar{K}$  until a nonsingular  $T$  is obtained. Once the desired  $T$  is obtained, say after  $d$  trials, the gain matrix is obtained by solving  $KT = \bar{K}$ , using Gaussian elimination for this purpose, about  $\left(\frac{1}{3}n^3 + \frac{n^2}{2}\right)$  flops are needed. This will bring the operations count that the algorithm

uses to the order of  $\left[d\left(\frac{109}{6}n^3 + 3n^2p\right) + \frac{1}{3}n^3 + \frac{n^2}{2}\right]$ .

We have to mention that the observability of the pair  $(F, \bar{K})$  has to be checked each time, and, this process is costly as well.

### 6.2.2 The Algorithm for Jordan Form

In this case we choose an  $n \times n$  matrix  $F$  in Jordan form and a corresponding  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  is observable. We then reduce  $A$  to Hessenberg form  $H = U^T A U$  and set  $S = F^T$ . We form the matrix  $G = U^T B \bar{K}$  and solve the system  $HY - YS^T = G$  for the intermediate solution  $Y$ . The solution  $T$  is then obtained by evaluating  $T = UY$  and the feedback matrix  $K$  is obtained by solving  $KT = \bar{K}$ , since  $T$  is nonsingular.

**The cost:** The transformation of  $A$  requires  $\left(\frac{5}{3}n^3\right)$  flops, and to form the matrix  $G$ ,  $(2n^2p)$  flops are needed. Concerning the computation of  $Y$ , the operation count is considerably reduced compared to the previous case, since, if all the eigenvalues of  $F$  are complex, which is the worst case, it will consist of a block diagonal matrix with  $2 \times 2$  blocks along the diagonal and all the other elements will be zeros, then the system will be solved using Gaussian elimination with partial pivoting in about  $(6n^2)$  operations. The computation of the solution  $T$  necessitates  $(n^3)$  flops, and computation of  $K$  is done using Gaussian elimination requiring the order of  $\left(\frac{1}{3}n^3 + \frac{1}{2}n^2\right)$ .

Hence, the order of the overall workcount of the algorithm to compute the feedback gain matrix is of  $\left(3n^3 + 2n^2p + \frac{13}{2}n^2\right)$  flops.

### 6.2.3 The Algorithm for Controller Form

Construct the  $n \times n$  matrix  $F$  in controller form and an arbitrary  $p \times n$  matrix  $\bar{K}$  such that  $(F, \bar{K})$  is observable, then reduce  $A$  and  $F$  respectively to Hessenberg and Schur forms  $H = U^T A U$  and  $S = V^T F^T V$ . Form the matrix  $G = U^T B \bar{K} V$  and solve the system  $HY - YS^T = G$  for the intermediate solution  $Y$ . The solution  $T$  is then obtained by

$T = UYV^T$ . The feedback matrix  $K$  is obtained by solving  $KT = \bar{K}$  since for this case also the proposed  $F$  guaranties the nonsingularity of  $T$ .

**The cost:**  $\left(\frac{5}{3}n^3\right)$  flops are used here also for the transformation of  $A$ . The reduction of  $F^T$  to Schur form uses  $\left(10n^3 - \frac{5}{3}n^3\right)$  flops since it is in Hessenberg form. To form the matrix  $G$ ,  $3n^2p$  flops are used by the algorithm. The determination of  $Y$  uses at most about  $\left(\frac{7}{2}n^3\right)$  flops. The solution  $T$  is obtained in  $2n^3$  flops. And finally the computation of  $K$ , and this requires  $\left(\frac{1}{3}n^3 + \frac{1}{2}n^2\right)$  flops.

The summation of these counts gives the order of the total operations count used by the algorithm for the controller form, which is of  $\left(\frac{95}{6}n^3 + 3n^2p + \frac{1}{2}n^2\right)$  flops.

### 6.2.3 The Block Form Algorithm

Construct the  $n \times n$  matrix  $F$  in block diagonal form (controller or Jordan form). Let  $\mu$  be the number of blocks with  $n_k$  the dimension of the  $k^{\text{th}}$  block. Let  $\bar{K}$  be an arbitrary  $p \times n$  matrix such that  $(F, \bar{K})$  is observable. Reduce  $A$  to Hessenberg form such that  $A = UHU^T$ . We have to solve in this case  $\mu$  times the equation  $AT_k - T_k F_k = B\bar{K}_k$  for  $T_k$ . First, reduce  $F_k$  to Schur form, such that  $F_k = V_k S_k^T V_k^T$ . The intermediate solution  $Y_k$  can then be obtained by solving the system of the form  $HY_k - Y_k S_k^T = G_k$  with  $G_k = U^T B\bar{K}_k V_k$ . Obtain the partial solution  $T_k = UY_k V_k^T$  the  $k^{\text{th}}$  block of the final solution. Form  $T$  and solve  $KT = \bar{K}$  for the gain matrix  $K$ .

**The cost:** The transformation of  $A$  to Hessenberg form uses  $\left(\frac{5}{3}n^3\right)$  flops. For every block  $F_k$ , reduce  $F_k$  to Schur form  $V_k S_k^T V_k^T$  requiring at most  $\left(\frac{25}{3}n_k^3\right)$  flops (if  $F_k$  is in controller form). Once  $G_k$  is formed using  $(n^2 p + n_k^2 p + n n_k p)$  flops,  $Y_k$  will then be determined in  $\left(3n^2 n_k + \frac{1}{2} n n_k^2\right)$  flops, the  $k^{\text{th}}$  block of the solution is obtained as  $T_k = U Y_k V_k^T$  using  $(n^2 n_k + n n_k^2)$  flops. The following step is to compute the feedback matrix from  $KT = \bar{K}$  which requires about  $\left(\frac{1}{3}n^3 + \frac{1}{2}n^2\right)$  flops. If we want to compute the cost of the whole algorithm, then we will have to sum up the operations count for each  $k$ , which gives an order of  $\left[2n^3 + \frac{1}{2}n^2 + \sum_{k=1}^{\mu} \left(n^2 p + 4n^2 n_k + \frac{3}{2} n n_k^2 + n n_k p + \frac{25}{3} n_k^3 + n_k^2 p\right)\right]$ .

### 6.3 Perturbation Analysis

In this section we will study the perturbation analysis of our algorithm for which the main computations are concentrated in solving the Lyapunov equation.

Before starting the analysis, let us first make some observation about the sensitivity of the underlying problem  $\Phi(T) = C$  where  $C = B\bar{K}$ . This system of equations can be written as described in section 4.3 in the form

$$Pt = C \tag{6.1}$$

where

$$P = (I_n \otimes A) - (F^T \otimes I_n) \tag{6.2}$$

and

$$t = \text{vec}(T)$$

$$c = \text{vec}(C)$$

If  $P$  is ill-conditioned, then small changes in  $A$ ,  $F$ , and/or  $C$  can induce relatively large changes in the solution. In order to deduce the effect of this on the transformation  $\Phi$ , let us define a norm on the space of linear transformations from  $\mathfrak{R}^{n \times n}$  to  $\mathfrak{R}^{n \times n}$ :

$$f: \mathfrak{R}^{n \times n} \rightarrow \mathfrak{R}^{n \times n} \quad \|f\| = \max_{T \in \mathfrak{R}^{n \times n}} \frac{\|f(T)\|_F}{\|T\|_F}$$

where  $\|\cdot\|_F$  is the Frobenius norm defined by  $\|W\|_F^2 = \sum_{i,j} |w_{ij}|^2$ . Note that for the linear transformation  $\Phi$  defined by  $\Phi(T) = AT - TF$  we have

$$\|\Phi\| = \|P\|_2 < \|A\|_2 + \|F\|_2$$

where  $P$  is defined by (6.2). If  $\Phi$  is nonsingular, then

$$\|\Phi^{-1}\| = \left[ \min_{T \in \mathfrak{R}^{n \times n}} \frac{\|\Phi(T)\|_F}{\|T\|_F} \right]^{-1} = \|P^{-1}\|_2$$

Consider solving  $AT - TF = C$  on a computer with machine precision  $u$ . Rounding errors of order  $u\|A\|_F$ ,  $u\|F\|_F$  and  $u\|C\|_F$  will be present in  $A$ ,  $T$ , and  $C$  respectively, before any algorithm even begins execution. Therefore, a computed solution  $\hat{T}$  satisfies

$$(A + H)\hat{T} - \hat{T}(F + I) = (C + J) \quad (6.3)$$

where

$$\|H\|_F < u\|A\|_F \quad (6.4)$$

$$\|I\|_F < u\|F\|_F \quad (6.5)$$

$$\|J\|_F < u\|C\|_F \quad (6.6)$$

By applying standard linear system perturbation analysis, it is possible to establish the following result. Assume that  $AT - TF = C$ ,  $(A + H)\hat{T} - \hat{T}(F + I) = (C + J)$  and (6.4), (6.5), and (6.6) hold. If  $\Phi(Z) = AZ - ZF$  is nonsingular,  $C$  is nonzero, and

$$u[\|A\|_F + \|F\|_F]\|\Phi^{-1}\| < \frac{1}{2}, \quad (6.7)$$

then

$$\frac{\|T - \hat{T}\|_F}{\|T\|_F} < 4u[\|A\|_F + \|F\|_F]\|\Phi^{-1}\| \quad (6.7)$$

## 6.4 Numerical Implementation

We are given a controllable system of dimension 9 described by the matrices  $A$  and  $B$  given below:

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 & 0.3 & 0 & 0.5 & 1 & 1 \\ 1 & -1 & 1 & -1 & -0.4 & 0 & 0.3 & 0 & 1 \\ 0.5 & 1 & 0.2 & 1 & -4 & 2 & 1 & -1 & 1 \\ 7 & 3 & 1 & 0 & 2 & 0.3 & 0.24 & 1 & 0 \\ 0 & 2 & 1 & 1 & 3 & 1 & 0.5 & 2 & 1 \\ 1 & 0 & 0 & -2 & 1 & 3 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0.5 & 3 & 4 & 0 & 2 & 1 \\ -9 & -3 & 1 & 1 & -1 & -3 & 0.5 & 0.2 & 1 \\ 1 & 0 & 2 & 1 & -2 & 1 & 2 & 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0.3 \\ -2 & 0.6 & 0 \\ 0.5 & -1 & 1 \\ 0.4 & 0.7 & 0.8 \\ 1 & 6 & 3 \\ -3 & 1 & -0.5 \\ 0.65 & 0.54 & 0.21 \\ 0.65 & 0.3 & 8 \\ 0.85 & 0.21 & 0 \end{bmatrix}$$

We want the poles of our system to be the solutions of the following characteristic equation:

$$(s+10)^3(s+3)^2(s+12)^3(s+15)$$

### Example 1:

In this example, we construct the matrix  $F$  in Jordan form consisting of four Jordan blocks deduced straight away from the above equation.

$\bar{K}$  is chosen so that  $(F, \bar{K})$  is an observable couple, the condition  $\bar{K}$  has to satisfy is that its columns 1, 4, 6 and 9 are individually linearly independent i.e. all different from zero.

$$F = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 3 & 1 & 5 & 4 & 2 & 6 & 1 & 2 & 1 \\ 3 & 4 & 1 & 5 & 1 & 1 & 2 & 7 & 1 \\ 5 & 1 & 7 & 2 & 1 & 3 & 5 & 3 & 8 \end{bmatrix}$$

Solving the Lyapunov equation for  $T$  results in the following:

$$T = \begin{bmatrix} 0.3840 & 0.5821 & 0.1255 & 0.2312 & -0.8305 & 0.2592 & 0.2473 & 0.8959 & 0.0505 \\ -0.6588 & -0.2042 & -1.2940 & -6.9118 & -5.4225 & -1.1425 & -0.2410 & -0.1639 & -0.1384 \\ 1.9716 & 1.1269 & 2.7670 & 12.7961 & 8.9172 & 1.2540 & 1.3765 & 1.3673 & 1.2237 \\ -0.3400 & -0.6641 & -0.0540 & 0.1799 & 4.1501 & 0.1542 & -0.2252 & -0.7161 & -0.0038 \\ 2.2168 & 2.0690 & 1.9773 & 7.3135 & 3.4277 & 1.2889 & 1.6902 & 3.2342 & 1.2560 \\ -1.1002 & -0.3694 & -1.8461 & 0.2164 & 4.2331 & -1.4646 & -0.6057 & -0.5271 & -0.5792 \\ -0.7904 & -0.8304 & -0.6210 & -8.1091 & -11.3901 & 0.2041 & -0.6768 & -1.0068 & -0.5375 \\ 4.1031 & 1.8833 & 5.1612 & -0.2648 & -2.1988 & 1.8219 & 3.5858 & 3.2528 & 4.1864 \\ 0.1574 & 0.2829 & 0.0728 & 1.3343 & 1.3845 & 0.2626 & 0.0385 & 0.3782 & -0.0765 \end{bmatrix}$$

The above matrix is nonsingular, thus the gain matrix is determined by solving  $\bar{K} = KT$  and gives

$$K^T = \begin{bmatrix} 131.290314 & -81.084274 & -302.599535 \\ 136.895586 & -131.164970 & -524.185818 \\ 122.460482 & 108.421638 & -406.100294 \\ 78.61215 & -66.131712 & -263.761893 \\ -61.865625 & 49.167261 & 164.329203 \\ 4.026841 & 0.770172 & 37.282916 \\ 53.438641 & -46.606828 & -155.474627 \\ -3.511618 & 4.868738 & 32.044664 \\ 166.877039 & -164.851680 & -576.54806 \end{bmatrix}$$

We compute the new system's poles which are the eigenvalues of  $(A - BK)$  given by the column below:

$$\begin{bmatrix} -9.9600 + 0.068i \\ -9.9600 - 0.068i \\ -10.0798 \\ -2.9935 \\ -3.0065 \\ -12.0286 + 0.0551i \\ -12.0286 - 0.0551i \\ -11.9426 \\ -15.0003 \end{bmatrix}$$

### Example 2:

In this example, we will illustrate the result given by choosing  $F$  in controller form for the same characteristic equation as in example 1. For this purpose we develop the equation in order to determine the coefficients of  $s$ , we then construct the matrix for which the number of blocks is not unique, we choose for this case three blocks according to the given equation. and this results in the form of  $F$  given below, and a  $\bar{K}$  is selected to satisfy the observability condition.

$$(s^3 + 30s^2 + 300s + 1000)(s^3 + 36s^2 + 432s + 1728)(s^3 + 21s^2 + 99s + 135).$$

Thus  $F$  will be given by:

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1000 & -300 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1728 & -432 & -36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -135 & -99 & -21 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 3 & 1 & 5 & 4 & 2 & 6 & 1 & 2 & 1 \\ 3 & 4 & 1 & 5 & 1 & 1 & 2 & 7 & 1 \\ 5 & 1 & 7 & 2 & 1 & 3 & 5 & 3 & 8 \end{bmatrix}$$

The resulting Lyapunov equation is solved for the nonsingular  $T$  given by the following:

$$T = \begin{bmatrix} -31.0992 & -1.3287 & 0.2166 & -32.9166 & -4.4348 & 0.1090 & 3.9392 & 7.2698 & 0.4670 \\ -38.9032 & 8.3556 & -0.0085 & -31.3252 & 10.6448 & -0.0125 & 31.5056 & 16.4873 & 0.8243 \\ 72.9570 & -21.9155 & -0.4385 & 70.1397 & -10.7519 & -0.1595 & -58.0775 & -47.7054 & -1.7431 \\ 56.3354 & 9.6127 & 0.5341 & 36.6653 & 4.1735 & 0.2464 & 20.7667 & 3.0357 & 0.1297 \\ -41.1507 & -15.8625 & 0.4911 & -13.7108 & -13.8994 & 0.1864 & -53.4842 & -11.6314 & 0.5455 \\ 49.4698 & 24.9804 & 0.2247 & 37.3183 & 21.1307 & 0.0704 & 62.1947 & 21.4244 & 0.5930 \\ 0.7065 & 7.7053 & 0.3632 & 15.3535 & 0.4948 & 0.1368 & -108.9751 & -17.6524 & -1.1961 \\ -198.0761 & -71.3677 & -0.2997 & -152.3621 & -36.2968 & -0.1912 & 39.3890 & -15.5481 & 2.8316 \\ -36.8438 & -3.4111 & 0.0443 & -25.4589 & -5.0498 & 0.0111 & 5.8306 & 10.6029 & 0.5844 \end{bmatrix}$$

The feedback matrix is determined by solving  $\bar{K} = KT$  and is such that:

$$K^T = \begin{bmatrix} 599.0913052 & 3.3948146 & -153.348713 \\ 80.9833095 & 16.6169974 & 15.2950893 \\ 409.1519306 & 6.7263829 & -94.1145062 \\ 43.7258729 & 7.7426892 & 16.3727803 \\ -329.9886814 & 1.5651787 & 88.0313980 \\ 370.0210366 & -7.6809119 & -112.6630397 \\ 275.8297275 & -2.9897447 & -73.0893747 \\ 75.2703311 & -2.9599099 & -22.6840250 \\ 751.7884247 & 8.4597020 & -177.2296445 \end{bmatrix}$$

We then verify that eigenvalues of  $(A - BK)$  are the desired poles and they are given by:

$$\begin{bmatrix} -3.0000 + 0.0037i \\ -3.0000 - 0.0037i \\ -15.0000 \\ -9.9535 + 0.0805i \\ -9.9535 - 0.0805i \\ -10.0929 \\ -11.9482 + 0.0904i \\ -11.9482 - 0.0904i \\ -12.1036 \end{bmatrix}$$

The examples to follow are to illustrate the block algorithm for both the Jordan and controller forms, for this purpose, let us consider the controllable system described by:

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 & 0.3 & 0 & 0.5 & 1 & 1 & -0.61 \\ 1 & -1 & 1 & -1 & -0.4 & 0.3 & 1 & 0.3 & -0.04 & 0.09 \\ 0.5 & 1 & 1 & -0.5 & 1 & -4 & 2 & 1 & -1 & 1 \\ -1 & 7 & 3 & 1 & 0 & 2 & 0.3 & 0.24 & 1 & 0 \\ 1 & -0.9 & 0.5 & -0.8 & 1 & 3 & 1 & 0.5 & 2 & 1 \\ -0.9 & -0.5 & -0.08 & 0 & -2 & 1 & 3 & 2 & 1 & 0 \\ 2 & 1 & 0.63 & -0.5 & -0.7 & 1 & 0.5 & 3 & 4 & 1 \\ -9 & -3 & 1 & -0.5 & 0.29 & 1 & -1 & -3 & 0.5 & 0.2 \\ 0.6 & -0.8 & 0.9 & 1.5 & 0.6 & 0.19 & 0.5 & 1 & 0.6 & 0.32 \\ -0.87 & 2 & -1 & 0.65 & -3 & 0.54 & 0.8 & 0.81 & 0.35 & -0.9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 0.3 \\ 0.5 & -1 & 1 \\ 0.4 & 0.7 & 0.8 \\ 1 & 6 & 3 \\ 0 & 0.6 & -0.65 \\ -3 & 1 & -0.5 \\ 0.65 & 0.54 & 0.21 \\ -0.32 & 0 & -1 \\ 0.65 & 0.3 & 8 \\ -0.95 & 0.32 & 3 \end{bmatrix}$$

We want our system to have two real poles -10 and -8.5 both with multiplicity 3 and two pairs of complex poles:  $(-7 \pm 3i)$  and  $(-6 \pm 4i)$ .

**Example 3:**

In this example, we construct  $F$  in Jordan form which consists of four blocks, for each one of these blocks corresponds a block in  $\bar{K}$ , for each pair  $(F_i, \bar{K}_i)$  we solve a Lyapunov equation for  $T_i$  this results in the following four steps:

**Step 1:**

$$F_1 = \begin{bmatrix} -10 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & 0 & -10 \end{bmatrix}, \quad \bar{K}_1 = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 4 & 3 \\ 5 & 1 & 3 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0.5174 & 0.4937 & 0.6494 \\ 0.8056 & 0.0066 & 0.5612 \\ 0.7526 & 0.5568 & 0.4324 \\ 2.0384 & 2.5558 & 2.7549 \\ -0.4209 & 0.0798 & 0.0839 \\ -1.1216 & -0.0583 & -1.2895 \\ -1.2071 & -0.3280 & -0.5586 \\ -0.1464 & 0.4199 & 0.5384 \\ 3.8388 & 0.8156 & 2.2928 \\ 1.0099 & 0.3912 & 0.4222 \end{bmatrix}$$

**Step 2:**

$$F_2 = \begin{bmatrix} -8.5 & 1 & 0 \\ 0 & -8.5 & 1 \\ 0 & 0 & -8.5 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} 4 & 2 & 8 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.3981 & 0.3075 & 0.7250 \\ 0.6308 & 0.5431 & 0.8908 \\ 0.0945 & 0.3640 & -0.5500 \\ 1.3115 & 1.0916 & 2.5095 \\ 0.0802 & -0.1308 & 0.8416 \\ -1.2879 & -0.7082 & -2.6183 \\ -0.5681 & -0.8855 & -0.0909 \\ 0.4512 & 0.2132 & 1.7164 \\ 1.8782 & 1.9856 & 1.2431 \\ 0.1598 & 0.4272 & -0.6156 \end{bmatrix}$$

Step 3:

$$F_3 = \begin{bmatrix} -7 & 3 \\ -3 & -7 \end{bmatrix}, \quad \bar{K}_3 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 3 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0.1930 & 1.1051 \\ 0.4598 & 0.6076 \\ 0.4675 & 2.1288 \\ 0.8316 & 4.7735 \\ -0.7580 & -0.6067 \\ -0.3767 & -0.3122 \\ -0.7006 & -2.3698 \\ -1.1845 & 0.0708 \\ 3.8465 & 4.0906 \\ 1.2165 & 1.5159 \end{bmatrix}$$

Step 4:

$$F_4 = \begin{bmatrix} -6 & 4 \\ -4 & -6 \end{bmatrix}, \quad \bar{K}_4 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0.0533 & 0.5883 \\ 0.9214 & 1.0603 \\ -0.0723 & 1.6698 \\ 1.8650 & 1.2650 \\ -0.4711 & -0.9194 \\ -1.3992 & -0.1864 \\ -0.4719 & -3.2067 \\ -0.6415 & -0.5191 \\ 4.7994 & 4.2211 \\ 1.3703 & 1.2276 \end{bmatrix}$$

The solution column blocks are gathered to form  $T$  used to compute the gain matrix:

$$K^T = \begin{bmatrix} 32.0669003 & 22.7484070 & 1.6606698 \\ 45.8023719 & 7.4868903 & 10.4287272 \\ -16.8178654 & -12.2900835 & 6.8660052 \\ 3.8974952 & -0.0532794 & -1.4674688 \\ -81.0950976 & -11.4803739 & -0.9818741 \\ 4.1231936 & 0.2836504 & 5.7614295 \\ 16.1308862 & -0.6290900 & 9.6179223 \\ 15.6953001 & 0.4586815 & 4.2308909 \\ -14.9332334 & -7.0685402 & 4.1047707 \\ 4.7606755 & 15.3333805 & -3.8776950 \end{bmatrix}$$

The eigenvalues of  $(A - BK)$  are:

$$\begin{bmatrix} -6.0000 + 4.0000i \\ -6.0000 - 4.0000i \\ -7.0000 + 3.0000i \\ -7.0000 - 3.0000i \\ -8.5006 + 0.0011i \\ -8.5006 - 0.0011i \\ -8.4987 \\ -10.0008 + 0.0013i \\ -10.0008 - 0.0013i \\ -9.9985 \end{bmatrix}$$

#### Example 4:

This example will be for the same set of desired poles but we are changing the number of blocks. The solution in this case will be determined in six steps.

#### Step 1:

$$F_1 = \begin{bmatrix} -10 & 1 \\ 0 & -10 \end{bmatrix}, \quad \bar{K}_1 = \begin{bmatrix} 4 & 1 \\ 2 & 4 \\ 5 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.5174 & 0.4937 \\ 0.8056 & 0.0066 \\ 0.7526 & 0.5568 \\ 2.0384 & 0.5558 \\ -0.4209 & 0.0798 \\ -1.1216 & -0.0583 \\ -1.2071 & -0.3280 \\ -0.1464 & 0.4199 \\ 3.8388 & 0.8156 \\ 1.0099 & 0.3912 \end{bmatrix}$$

Step 2:

$$F_2 = [-10], \quad \bar{K}_2 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.6290 \\ 0.5416 \\ 0.3695 \\ 2.5539 \\ 0.0725 \\ -1.2911 \\ -0.4830 \\ 0.4535 \\ 2.2555 \\ 0.3877 \end{bmatrix}$$

Step 3:

$$F_3 = \begin{bmatrix} -8.5 & 1 \\ 0 & -8.5 \end{bmatrix}, \quad \bar{K}_3 = \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0.3981 & 0.3075 \\ 0.6308 & 0.5431 \\ 0.0945 & 0.3640 \\ 1.3115 & 1.0916 \\ 0.0802 & -0.1308 \\ -1.2879 & -0.7082 \\ -0.5681 & -0.8855 \\ 0.4512 & 0.2132 \\ 1.8782 & 1.9856 \\ 0.1598 & 0.4272 \end{bmatrix}$$

Step 4:

$$F_4 = [-8.5], \quad \bar{K}_4 = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0.6947 \\ 0.8005 \\ -0.6324 \\ 2.4945 \\ 0.8719 \\ -2.5875 \\ 0.1578 \\ 1.6522 \\ 1.0220 \\ -0.6605 \end{bmatrix}$$

Step 5:

$$F_5 = \begin{bmatrix} -7 & 3 \\ -3 & -7 \end{bmatrix}, \quad \bar{K}_5 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 3 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 0.1930 & 1.1051 \\ 0.4598 & 0.6076 \\ 0.4675 & 2.1288 \\ 0.8316 & 4.7735 \\ -0.7580 & -0.6067 \\ -0.3767 & -0.3122 \\ -0.7006 & -2.3698 \\ -1.1845 & 0.0708 \\ 3.8465 & 4.0906 \\ 1.2165 & 1.5159 \end{bmatrix}$$

Step 6:

$$F_6 = \begin{bmatrix} -6 & 4 \\ -4 & -6 \end{bmatrix}, \quad \bar{K}_6 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 0.0533 & 0.5883 \\ 0.9214 & 1.0603 \\ -0.0723 & 1.6698 \\ 1.8650 & 1.2650 \\ -0.4711 & -0.9194 \\ -1.3992 & -0.1864 \\ -0.4719 & -3.2067 \\ -0.6415 & -0.5191 \\ 4.7994 & 4.2211 \\ 1.3703 & 1.2276 \end{bmatrix}$$

Gathering the six column blocks of  $T$  to form the final solution required to compute the feedback matrix  $K$ .

$$K^T = \begin{bmatrix} 32.0669003 & 22.7484070 & 1.6606698 \\ 45.8023720 & 7.4868903 & 10.4287272 \\ -16.8178654 & -12.2900835 & 6.8660052 \\ 3.8974952 & -0.0532794 & -1.4674688 \\ -81.0950976 & -11.4803739 & -0.98187410 \\ 4.1231936 & 0.2836505 & 5.7614295 \\ 16.1308862 & -0.6290899 & 9.6179223 \\ 15.6953001 & 0.4586815 & 4.2308909 \\ -14.9332334 & -7.0685402 & 4.1047707 \\ 4.7606755 & 15.3333805 & -3.8776950 \end{bmatrix}$$

We verify once again that the poles of the new system are the ones desired.

$$\begin{bmatrix} -6.0000 + 4.0000i \\ -6.0000 - 4.0000i \\ -7.0000 + 3.0000i \\ -7.0000 - 3.0000i \\ -10.0000 + 0.0030i \\ -10.0000 + 0.0030i \\ -10.0000 \\ -8.5040 \\ -8.4960 \\ -8.5000 \end{bmatrix}$$

### Example 5:

For the controller form, the coefficients of the characteristic equation have to be determined. We decide first on the number of blocks, if for instance we choose  $F$  to have four blocks, one possible form of the characteristic equation can be the following:

$$(s^3 + 30s^2 + 300s + 1000)(s^3 + 25.5s^2 + 216.75s + 614.125)(s^2 + 14s + 58)(s^2 + 12s + 52).$$

Hence, the solution of the Lyapunov equation will be obtained in four steps:

#### Step 1:

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1000 & -300 & -30 \end{bmatrix}, \quad \bar{K}_1 = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 4 & 3 \\ 5 & 1 & 3 \end{bmatrix}, \quad T_1 = \begin{bmatrix} -9.4293 & -4.5517 & 0.2279 \\ 40.7373 & -1.3157 & 0.0212 \\ 9.6831 & 0.2407 & 0.2607 \\ -122.6831 & -37.8236 & -0.2298 \\ 5.2206 & -0.7848 & -0.0769 \\ 44.2570 & 19.5779 & 0.2129 \\ -132.8579 & -10.8136 & -0.2444 \\ 80.0196 & 4.1717 & 0.0277 \\ -7.9776 & -20.8841 & 0.2400 \\ -11.2494 & -4.4286 & 0.0375 \end{bmatrix}$$

Step 2:

$$F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -614.125 & -216.75 & -25.5 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} 4 & 2 & 8 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -35.0413 & -8.8144 & 0.1152 \\ 84.2576 & 7.0369 & 0.4208 \\ -124.1198 & -10.8660 & -0.2346 \\ -47.6928 & -28.3460 & -0.2835 \\ 94.2513 & 5.6935 & 0.2569 \\ -3.7021 & 20.5136 & -0.0704 \\ -105.8904 & -20.2182 & -0.6832 \\ 293.9198 & 31.8974 & 1.3366 \\ -46.0862 & -12.0002 & 0.0639 \\ -55.7534 & -3.1999 & -0.3131 \end{bmatrix}$$

Step 3:

$$F_3 = \begin{bmatrix} 0 & 1 \\ -58 & -14 \end{bmatrix}, \quad \bar{K}_3 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 3 \end{bmatrix}, \quad T_3 = \begin{bmatrix} -1.4785 & 0.6749 \\ -4.8940 & -0.4747 \\ -4.9340 & 0.2723 \\ -24.8650 & 1.6836 \\ -4.4752 & -0.3887 \\ 2.5180 & -0.2569 \\ 10.4441 & 0.8882 \\ -17.1598 & -1.4134 \\ -1.2535 & 1.6481 \\ -2.2953 & 0.4061 \end{bmatrix}$$

Step 4:

$$F_4 = \begin{bmatrix} 0 & 1 \\ -52 & -12 \end{bmatrix}, \quad \bar{K}_4 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} -0.0942 & 0.2995 \\ 0.4605 & 0.0206 \\ -0.3603 & 0.3312 \\ -7.6541 & 0.5146 \\ -2.6866 & -0.3157 \\ -0.3161 & -0.1063 \\ -0.8379 & -0.2762 \\ -5.9303 & -0.5928 \\ 6.9636 & 1.2556 \\ 1.1907 & 0.3125 \end{bmatrix}$$

Having obtained the four block columns of  $T$ , the feedback matrix is then computed:

$$K^T = \begin{bmatrix} 24.2189360 & 19.70049551 & -3.7625208 \\ 22.5564255 & -3.8830668 & 17.4506296 \\ -1.3098697 & -7.0013589 & 13.8222099 \\ -1.8718114 & -2.3644269 & -0.9934627 \\ -30.4169246 & -2.9473932 & 20.4417531 \\ 2.0577639 & -1.5780807 & 11.4795606 \\ 12.1030658 & -0.7912750 & 10.6111164 \\ 9.8229054 & 3.0460920 & -3.5316795 \\ -5.7261300 & -3.5641275 & 8.1714125 \\ 5.2863167 & 14.9706277 & -12.9518420 \end{bmatrix}$$

We compute the new system's poles and are given by the column below:

$$\begin{bmatrix} -6.0000 + 4.0000i \\ -6.0000 - 4.0000i \\ -7.0000 + 3.0000i \\ -7.0000 - 3.0000i \\ -8.4999 + 0.0002i \\ -8.4999 - 0.0002i \\ -10.0001 + 0.0002i \\ -10.0001 - 0.0002i \\ -8.5002 \\ -9.9998 \end{bmatrix}$$

### Example 6:

In this example, we shall see the results for  $F$  in controller form consisting of only two blocks on the diagonal. We develop the characteristic equation and one possible form is given by:

$$(s^5 + 37.5s^4 + 574.75s^3 + 4541.125s^2 + 18640.5s + 31934.5)(s^5 + 44s^4 + 778s^3 + 6940s^2 + 31400s + 58000).$$

The determination of the solution  $T$  is done in two steps:

#### Step 1:

$$F_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -31934.5 & -18640.5 & -4541.125 & -574.75 & -37.5 \end{bmatrix}, \quad \bar{K}_1 = \begin{bmatrix} 4 & 1 & 5 & 4 & 2 \\ 2 & 4 & 3 & 1 & 1 \\ 5 & 1 & 3 & 2 & 2 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0.0671 & -0.0656 & -0.0198 & -0.0030 & 0.0000 \\ -0.9681 & -0.3037 & -0.0304 & -0.0047 & -0.0001 \\ 1.7033 & 0.4085 & 0.0590 & 0.0023 & 0.0002 \\ -0.0959 & 0.0983 & -0.0304 & -0.0125 & 0.0000 \\ -1.1353 & -0.2955 & -0.0378 & -0.0013 & -0.0001 \\ 0.5877 & 0.1125 & 0.0399 & 0.0087 & 0.0001 \\ -0.1743 & 0.0955 & -0.0241 & 0.0020 & 0.0001 \\ -2.6087 & -0.9692 & -0.1420 & -0.0110 & -0.0004 \\ 0.1140 & -0.0818 & -0.0394 & -0.0167 & 0.0000 \\ 0.8373 & 0.2599 & 0.0312 & -0.0013 & 0.0001 \end{bmatrix}$$

Step 2:

$$F_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -58000 & -31400 & -6940 & -778 & -44 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} 8 & 1 & 3 & 2 & 1 \\ 2 & 2 & 6 & 1 & 2 \\ 1 & 5 & 3 & 6 & 1 \end{bmatrix}$$

$$T_2 = 10^3 \times \begin{bmatrix} -0.4849 & -0.1906 & -0.0418 & -0.0051 & 0.0000 \\ 0.8009 & 0.2887 & 0.0414 & 0.0030 & 0.0001 \\ -1.5546 & -0.3912 & -0.0563 & -0.0057 & -0.0001 \\ 0.3781 & -0.0520 & -0.0437 & -0.0160 & 0.0000 \\ 0.9768 & 0.2600 & 0.0471 & 0.0027 & 0.0001 \\ -0.7476 & -0.1669 & -0.0092 & 0.0008 & 0.0000 \\ -0.3491 & -0.3214 & -0.0495 & -0.0025 & 0.0000 \\ 2.5049 & 0.7665 & 0.1297 & 0.0066 & 0.0001 \\ -0.5262 & -0.1946 & -0.0903 & -0.0096 & 0.0000 \\ -0.7324 & -0.2531 & -0.0507 & -0.0047 & 0.0000 \end{bmatrix}$$

Now that the two column blocks solutions are determined  $T$  is constructed to compute the (3x10) gain matrix whose transpose is the following:

$$K^T = \begin{bmatrix} -218.2282483 & 76.1568743 & -219.4165566 \\ -58.6168021 & -67.6789901 & -58.7615177 \\ 50.8708746 & 12.7635938 & 51.3270017 \\ 14.5546194 & -21.8822976 & 14.1614533 \\ 96.8926448 & 100.2645853 & 97.5824270 \\ 144.3407975 & -79.7067698 & 144.3178144 \\ 49.9364612 & -43.4443926 & 50.3099797 \\ 34.81807110 & -29.4583183 & 35.0233043 \\ 03.4189876 & -10.3579948 & 103.9501211 \\ .17.2521673 & -20.3325463 & -16.6997208 \end{bmatrix}$$

The eigenvalues of  $(A - BK)$  are the following:

$$\begin{bmatrix} -6.0000 + 4.0000i \\ -6.0000 - 4.0000i \\ -7.0000 + 3.0000i \\ -7.0000 - 3.0000i \\ -8.4986 + 0.0023i \\ -8.4986 - 0.0023i \\ -8.5027 \\ -10.0031 \\ -9.9985 + 0.0026i \\ -9.9985 - 0.0026i \end{bmatrix}$$

In the examples to follow, we shall demonstrate the cases for which a solution  $K$  can not be obtained.

*Case where  $(A, B)$  is not controllable:*

A necessary condition for  $T$  to be nonsingular is that  $(A, B)$  controllable and  $(F, \bar{K})$  observable. So, let us now consider a non-controllable system and investigate the results of the algorithm for both the Jordan form and controller forms. Let the system be described by the following matrices  $A$  and  $B$ :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The rank of  $B$  is 2; hence, the system is controllable if and only if the matrix

$$[B : AB : A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix}$$

has rank 4. This is not the case; therefore, the system is not controllable.

**Example 7:**

$$F = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & -2 \end{bmatrix}$$

The intermediate solution  $Y$  is computed which is such that the final solution  $T = UYV^T$ ,

$$Y = \begin{bmatrix} 0.0148 & 0.0385 & 0.0056 & -0.0200 \\ 0.0355 & 0.1923 & 0.0192 & -0.1400 \\ 0.0296 & 0.0769 & 0.0112 & -0.0400 \\ 0.5510 & 0.7846 & -0.2677 & -0.1371 \end{bmatrix}$$

The third row of  $Y$  is the multiple of the first one so,  $\text{rank}(Y)=3$ . Hence, the solution is singular.

**Example 8:**

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -49 & -14 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & -2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.1131 & 0.0075 & 0.0594 & -0.0028 \\ 0.3696 & 0.0377 & 0.2744 & -0.0198 \\ 0.2263 & 0.0151 & 0.1188 & -0.0057 \\ -0.4218 & -0.2384 & 2.2429 & -0.3022 \end{bmatrix}$$

Here again, the third row of  $Y$  is the multiple of the first one so,  $\text{rank}(Y)=3$  which makes the solution singular.

*Case where  $(F, \bar{K})$  is not observable:*

In the following two examples we will explore the case where  $(F, \bar{K})$  is not observable. We consider then a controllable system described by the following pair  $(A, B)$ :

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The rank of  $B$  is 2; hence, the system is controllable if and only if the matrix

$$[B : AB : A^2B : A^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 3 & 4 & 9 & 3 \\ 1 & 0 & 2 & 1 & 1 & 2 & 12 & 10 \\ 0 & 0 & 1 & 3 & 8 & 1 & 22 & 8 \\ 1 & 0 & -2 & 1 & -2 & -2 & 5 & -2 \\ 0 & 1 & 2 & 0 & 7 & 2 & 11 & 16 \end{bmatrix}$$

has rank 5, this is verified, hence; our system is controllable.

### Example 9:

We select a matrix  $F$  in Jordan form with one block corresponding to a pair of conjugate complex eigenvalues and two blocks corresponding to a real eigenvalue as follows:

$$F = \begin{bmatrix} -6 & -4 & 0 & 0 & 0 \\ 4 & -6 & 0 & 0 & 0 \\ 0 & 0 & -8 & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$$

The condition  $\bar{K}$  has to satisfy for  $(F, \bar{K})$  to be observable, is that its first column has to be nonzero, and the first columns of the second and third blocks linearly independent. If we take

$$\bar{K} = \begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 3 \end{bmatrix},$$

Notice that the third and fifth columns of  $\bar{K}$  are linearly dependent, hence, the observability condition is not satisfied. We solve the Lyapunov equation for this case and we get:

$$Y = \begin{bmatrix} -0.0214 & 0.0060 & -0.0035 & 0.0002 & -0.0002 \\ -0.1539 & -0.0009 & 0.0318 & 0.3430 & -0.3430 \\ 0.0927 & 0.0149 & -0.0345 & -0.1744 & 0.1744 \\ -0.2114 & 0.0431 & 0.0221 & 0.2238 & -0.2238 \\ -0.0546 & 0.1182 & -0.0277 & -0.1368 & 0.1368 \end{bmatrix}$$

which is not full rank since the fourth and fifth columns are linearly dependent, therefore,  $Y$  is singular.

#### Example 10:

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -52 & -12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -64 & -16 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$$

For  $(F, \bar{K})$  to be observable,  $\bar{K}$  has to be chosen such that its three column blocks corresponding to the three blocks in  $F$  are all non zero, if we take for instance the elements

of the last column all zeros, the observability condition will not be satisfied. For instance if

$$\bar{K} = \begin{bmatrix} 1 & 1 & 2 & 2 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{bmatrix},$$

we obtain

$$Y = \begin{bmatrix} 0.1414 & -0.0094 & -0.1514 & -0.0148 & 0 \\ -0.0245 & 0.0512 & -0.3236 & -0.1407 & 0 \\ -0.2879 & -0.0179 & 0.8455 & 0.1074 & 0 \\ 1.0185 & -0.0104 & -1.7071 & -0.2228 & 0 \\ 0.7101 & 0.0093 & -0.5952 & -0.0935 & 0 \end{bmatrix}$$

For this example, the computed matrix  $Y$  is singular since its fifth column is null.

#### Example 11:

This example illustrates the case where  $F$  does not satisfy the condition of not having common eigenvalues with those of  $A$ . Let the controllable pair be described by:

$$A = \begin{bmatrix} -3 & 1 & 1 & -1 & 4 \\ 0 & -3 & 2 & 3 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The poles of the original system are:

$$\begin{bmatrix} -3.0000 \\ -3.0000 \\ 2.8312 \\ -0.4156 + 0.4248i \\ -0.4156 - 0.4248i \end{bmatrix}$$

If we choose  $F$  and  $\bar{K}$  as follows:

$$F = \begin{bmatrix} -6 & -4 & 0 & 0 & 0 \\ 4 & -6 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix} \quad \bar{K} = \begin{bmatrix} 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

For the above pair of arbitrary matrices, the  $Y$  is undefined, its entries are NaN which, in MATLAB, stands for Not-a-Number obtained as a result of mathematically undefined operations like  $0.0/0.0$  and  $\text{inf}-\text{inf}$ .

Multivariable design through state feedback, being a well-established approach, is a widely used method. This popularity is also mainly due to its simplicity and the quality of control obtained. Within this scope, many research papers have been written in this subject, therefore, different algorithms have been proposed. One of these is Chen's algorithm that we have used in this thesis. It allows the designer to preset the form of the compensated system, therefore preset the closed-loop eigenstructure. It is required in this method to choose an observable pair of matrices, and the feedback matrix is obtained after solving a Lyapunov equation whose solution must be non-singular, if not the algorithm has to be repeated for a different pair of matrices.

The contribution of this work is to propose forms for the arbitrary matrices that will make this algorithm converge at first execution as well as optimising the overall cost of the procedure. Consequently, we have first selected an efficient method for solving the resulting Lyapunov equation, then investigated among the canonical forms those for which the observability is ensured and for which the Lyapunov equation is more effectively solved.

In this context, the Hessenberg Schur algorithm is used for solving the equation. This method is based on orthogonal decomposition, hence, the condition of the problem is not changed.

Concerning the forms of the matrices, the Jordan and controller canonical forms have been judged suitable for our design. According to these forms we have two approaches. The Jordan form approach and the controller form approach. We have further extended this algorithm to handle block matrices, which gives the third approach, where the original equation has been decomposed into a set of equations of smaller dimensions.

This choice is based on the fact that the main steps of the Hessenberg Schur algorithm is the reduction of the first and second matrices in the obtained Lyapunov equation to, respectively, Hessenberg and Schur forms using similarity transformations. In our case, the first matrix is the open-loop system's matrix and the second one is the arbitrary one whose eigenvalues are the desired poles.

In the first approach, we propose to take our matrix in Jordan form. This form enables us to reduce the number of operations of the Hessenberg Schur algorithm by about  $(11n^3 + n^2 p)$  since our matrix does not need to be reduced to Schur form, and the multiplication by the corresponding similarity transformation is skipped twice.

The non-singularity of the solution is ensured due to the fact that every matrix is similar to a matrix in Jordan form, and the solution, which is the similarity transformation, is therefore non-singular.

For the second approach, we take our arbitrary matrix in controller form, the number of operations of Hessenberg Schur algorithm is reduced by about  $\left(\frac{5}{3}n^3\right)$  flops which is the cost of reducing a n-dimensional matrix to Hessenberg form under which our arbitrary matrix is.

The nonsingularity of the solution is ensured here also because the controllability of a system is preserved by state feedback, hence, there exists a similarity transformation that transforms the compensated system's matrix into a controller form.

In the third approach, beside the time and space complexities that have been improved, because of the use of sub-matrices instead of the whole matrix, an important feature of this approach is that dealing with matrices of smaller dimensions introduce less perturbation to the problem.

Above, we have considered how the cost of the Hessenberg Schur algorithm has been reduced; if the whole algorithm is examined, we will see that from what has been proposed, two costly steps are omitted. These steps are the tests involved in checking the observability of the arbitrary pair of matrices and repeating it if this property is not satisfied, and the tests involved in checking the nonsingularity of the solution of the Lyapunov equation.

One may wonder why the observer form has not been proposed, as it presents the advantage that the observability condition is given by definition. The answer to this is that the nonsingularity of the solution is not insured. Even if the uncompensated system were observable, there is no guarantee that the closed-loop system will be so.

The block algorithm presented in this thesis determines the column blocks of solution of the Lyapunov equation, each one independently from the others. A suggestion for further research would be the introduction of parallel processing for this task. Therefore, determining these blocks simultaneously will considerably reduce the time required by the algorithm. An interesting point worth investigating is the possibility of finding a sufficient condition to obtain a non-singular solution to the Lyapunov equation for the general case.

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