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# THESIS

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## DEGREE OF MAGISTER

in Electronic Systems Engineering

by

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## *ON THE IRREDUCIBLE REALIZATION OF RATIONAL TRANSFER FUNCTION MATRICES*

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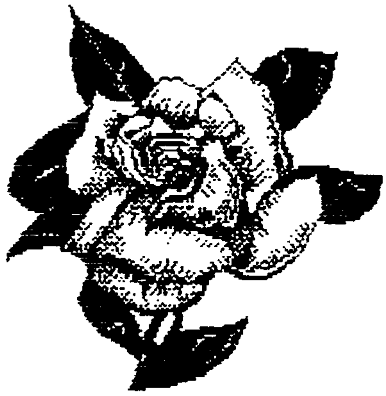

# DEDICACES

A mes chers parents

A ma famille

A Amine, Sofiane et Imene
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A tous ceux qui oeuvrent pour
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yaici

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ABSTRACT

The problem of finding the minimal realization of a given transfer function matrix has been studied by several authors. It is to determine state variable realizations with the minimum possible number of states. The irreducible realization is useful in analog and digital computer simulation or operational amplifier circuit synthesis of the transfer function. Irreducible realizations are desirable for reasons of economy and sensitivity.

The main contributions of this thesis can be summarized as follows:

1_ The development of a recursive algorithm, based on orthogonally projected vectors, that searches for linear independence in a set of vectors.

2_ The presentation of a method based on Chen's method [6-8] which uses the recursive algorithm instead of a row-searching algorithm.

3_ The presentation of a new method based on Kalman's method [15-17]. Starting from a controllable realization, easily obtained, we reduce it to a minimal one using similarity transformations.

4_ By duality, starting from an observable realization, a method is developed to reduce it to a minimal one.

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## SYMBOLS AND NOTATIONS

| | |
|---|---|
| $u$ | input variable. |
| $y$ | output variable. |
| $x_i$ | state variables. |
| $g$ | rational transfer function. |
| $\hat{g}$ | proper rational transfer function. |
| $G$ | rational transfer function matrix. |
| $\hat{G}$ | proper rational transfer function matrix. |
| $\Psi$ | least common denominator. |
| $h_i$ | Markov parameters. |
| $H(n,m)$ | Hankel matrix of order *nxm*. |
| $\{A,B,C,E\}$ | a realization of G. |
| $Cn(A,B)$ | Controllability matrix. |
| $O(C,A)$ | Observability matrix. |
| $\rho A$ | rank of a matrix A. |
| $A'$, $A^T$ | transpose of A. |
| $A^*$ | complex conjugate transpose of A. |
| $A^{-1}$ | inverse of A. |
| $A^+$ | pseudoinverse of A. |
| Adj A | adjoint matrix of A. |
| $I_n$ | *nxn* Identity matrix |

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# CHAPTER 1

## INTRODUCTION

Although the problems of automatic control were at first treated as problems of rational mechanics, and, as such, were tackled by time-domain methods, they soon came, particularly under the influence of electrical and electronic engineers, to be treated as filtering problems, i.e, by using frequency domain techniques.

The developments made along these lines, associated with the concept of a transfer function, have yield the single-input single-output methods and techniques. These frequential methods began, some years ago, gradually to give place to a theory of control in state-space, where the system to be controlled is represented by a vector differential equation of first order, in place of an operational transmittance.

But the increasing complexity of systems to be controlled and the necessity of a better representation, revealed that the use of frequency-domain methods leads to inexplicable contradictions. Such errors are most often due to a lack of appreciation of the ideas of controllability and observability. These notions are fundamental, specially in multivariable systems (in practice as well as in theory), both in problems of representation, structure and control.

Chen [6] proposed two methods based on Hankel matrices. The first one consists of searching a Hankel matrix using a row-searching algorithm and constructing the minimal realization from coefficients computed from this Hankel matrix. The second one uses the singular-value decomposition of a Hankel matrix to find an irreducible realization.

The second route is to employ the properties of polynomial matrices, and leads to Jordan's canonical realization [34-37], as well as controllable or observable companion forms of irreducible realization [15-17].

Some have developed coprime methods [6-8]. It consists of writing the transfer function matrix as a product of a numerator matrix by a denominator matrix which should be right coprime (left coprime) in case of controllable form (observable form) realization, then computing new matrices from the denominator to construct the minimal realization.

The third route is to write down the controllable or observable companion forms by inspection of the transfer function matrix and to obtain minimal realization by extracting the unobservable or uncontrollable parts. Kalman[15-17] gave a solution to the problem of obtaining a minimal realization based on similarity transformations determined from the controllability and observability matrices.

## 1.3. PRESENTATION OF THE THESIS

The purpose of this thesis is to present a first new method, based on Hankel matrices, which consists of a modified Chen's method and uses a recursive algorithm based on orthogonally projected vectors instead of the row-searching algorithm.

The second method consists of a simplified Kalman's reduction with the introduction of the recursive algorithm when needed.

The thesis is organized as follows:

In chapter 2, we will give a brief review on control theory starting from the controllability and observability principles which lead to minimal realization theory, then a section on Hankel theory.

Chapter 3 presents the two methods based on Hankel matrix already introduced earlier.

In chapter 4, we present Kalman's method.

Chapter 5 constitutes the heart of the thesis and represents the main contribution of this work. In this chapter, we present our methods which are the modified Chen's method and the modified Kalman's method.

In chapter 6, some numerical results are presented. We have tested our methods on examples used by some authors to illustrate their own methods and made a comparison.

Finally, in chapter 7, we provide a conclusion to this thesis and we suggest topics for further research.

# CHAPTER 2

# REALIZATION THEORY

We know that the properties of a physical process depend on a number of quantities which we can, in general, identify either as input variables u or output variables y. The relations defining the process, deduced from the laws of physics, are then of the form: $G(u,y)=0$.

We also know that these input-output relations do not, in general, enable us to determine uniquely the behavior of the system.

The following section gives the different forms to reperesent a system. A system can be controllable or observable and have a minimal realization, these concepts are discussed in the three next sections. The last section of this chapter deals with Hankel Theory.

## 2.1 STATE SPACE REPRESENTATION

Systems are generally represented in state space approach. The concept of state and state space representation is given in the following sections.

### 2.1.1 Mathematical representation of systems:

Many forms are possible, but generally, they divide into one of two categories: a)_ input-output equations and b)_ equations which reveal the internal behavior of the system as well as terminal characteristics.

Input-output equations are derived by a process of elimination of all system variables except those constituting inputs and those considered as outputs.

Models developed experimentally from measurements of inputs and outputs are almost invariably of the input-output type. Subsequent conversions to other forms, such as state variable models, are always possible.

When the constituent equations are linear with constant coefficients, Laplace or Z-transforms can be used to define input-output transfer functions. When more than one input and more than one output must be treated, matrix notation and the concepts of transfer matrices are convenient.

Integral forms of the input-output equations, using the system weighting function, are also widely used. The weighting function, or system impulse response, is obtained from the inverse Laplace or Z-transform of the input-output transfer function.

Equations which reveal the internal behavior of the system, and not just the terminal characteristics, can take several forms. To illustrate it let us take the example of circuit theory. Loop equations, involving all unknown loop currents (through variables), or node equations, involving all unknown nodal voltages (across variables), can be written. The choice between alternatives may depend on the number of loops versus the number of nodes. If everything is linear, transform techniques lead to the concepts of impedance and admittance matrices. Hybrid combinations of voltage and current equations are also used.

The state space approach will be developed extensively in the remainder of this thesis. At this time it is sufficient to say that state variables consist of some minimum set of variables which are essential for completely describing the internal status, i.e., state of the system.

### 2.1.2 The concept of state:

It is a complete summary of the status of the system at a particular point in time. Knowledge of the state at some initial time $t_0$, plus knowledge of the system inputs after $t_0$, allows the determination of the state at a later time $t_1$. As far as the state at $t_1$ is concerned, it makes no difference how the initial state was attained. Thus the state at $t_0$ constitutes a complete history of the system behavior prior to $t_0$, insofar as that history affects future behavior. At any fixed time the state of a system can be described by the values of a set of variables $x_i$, called state variables.

Only systems which can be described by a finite number n of state variables will be considered here. Then the state can be represented by an n component state vector $x=[x_1 \quad x_2 \quad \dots \quad x_n]$

**Definition2.1:**

The state variables of a system consist of a minimum set of parameters which completely summarize the system in the following sense. If at any time $t_0 \in T$, the values of the state variables $x_i(t_0)$ are known, then the output $y(t_1)$ and the values $x_i(t_1)$ can be uniquely determined for any time $t_1 \in T$, $t_1 > t_0$, provided $u_{[t_0,t_1]}$ is known.

T: set of all real numbers $t \in [t_0, t_f]$

$u_{[t_0,t_1]}$: segment of an input function over the set of times in $[t_0,t_1]_T$.

System analyses generally consist of two parts: quantitative and qualitative.

In the quantitative study, it is dealt with the search for the exact response of the system to certain input and initial conditions.

In the qualitative study, the general properties of a system are seeked. The following sections introduces two main qualitative properties of linear dynamical equations: Controllability and Observability.

## 2.2 CONTROLLABILITY OF LINEAR SYSTEMS

In this section, the concept of controllability of linear dynamical equation is introduced. More precisely, the *state* controllability of linear *state* equation is studied.

### 2.2.1 Definition of controllability:

Roughly speaking, controllability studies the possibility of steering the state from the input. If a dynamical equation is controllable, all the modes of the equation can be excited from the input.

The definition of state controllability is as follows:

**Definition 2.2:**

A linear system is said to be controllable at $t_0$ if it is possible to find some input function (or sequence in the discrete case) u(t), defined over $t \in T$, which will transfer the initial state $x(t_0)$ to the origin at some finite time $t_1 \in T, t_1 > t_0$. That is, there exists some input $u_{[t_0,t_1]}$, which gives $x(t_1) = 0$ at a finite $t_1 \in T$. If this is true for all initial times $t_0$ and all initial states $x(t_0)$, the system is completely controllable.

Some authors define another kind of controllability involving the output y(t) called *output controllability*, not needed in this case.

Complete controllability is obviously a very important property. If a system is not completely controllable, then for some initial states no input exists which can drive the system to the zero state. It would be meaningless to search for an optimal control in this case.

### 2.2.2 State-Controllability Problem:

A realization {A,B,C} or often just the pair {A,B} is said to be (state) controllable if we can always set up some desired state vector at $t = 0_+$, say $x(0_+)$, by using an impulse input in $\dot{x}(t) = Ax(t) + Bu(t)$

If we let $B = [b_1, \ldots, b_p]$ so that

$$\dot{x}(t) = Ax(t) + b_1 u_1(t) + \ldots + b_p u_p(t)$$

and if $u(t) = g_1 \delta(t) + \ldots + g_n \delta^{(n-1)}(t)$ where each $\{g_i\}$ is a p vector

then

$$\mathbf{x}(0_+) = \mathbf{x}(0_-) + \mathrm{Cn}[g_1', \ldots, g_n']'$$

where *Cn* is the *nxnp* (controllability matrix):

$$\mathbf{Cn} = [\mathbf{B} \quad \mathbf{AB} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{2.1}$$

If *Cn* has full rank, then we can find a combination of columns that will give any arbitrary n-vector $x(0_+) - x(0_-)$.

This combination will determine the vector $[g_1' \ldots g_n']'$ and thereby the appropriate input vector u(.). Therefore the following theorem is stated:

**Theorem 2.1**

A realization {A,B,C} is state controllable if and only if the *nxnp* controllability matrix $C_n(A,B)$ has full rank n.

**proof:** see reference [6-8]

The following corollary is very useful in computer implementation. The dimension of the controllability matrix is considerably reduced.

**Corollary 2.1**

A realization {A,B,C} is controllable if and only if the $n \times (n - \bar{p} + 1)p$ matrix

$$C_{n-\bar{p}} = \left[B \ AB \cdots A^{n-\bar{p}}B\right]$$

where $\bar{p}$ is the rank of B, has full rank n.

**proof:** see reference [6-8]

## 2.3 OBSERVABILITY OF LINEAR SYSTEMS

In this section, the concept of observability is introduced.

### 2.3.1 Definition of observability:

Dual to controllability, observability studies the possibility of estimating the state from the output. If a dynamical equation is observable all the modes of the equation are observed from the output.

**Definition 2.3:**

A linear system is said to be observable at $t_0$ if $x(t_0)$ can be determined from the output $y_{[t_0,t_1]}$ (or output sequence) for $t_0 \in T$ and $t_0 \le t_1$, where $t_1$ is some finite time belonging to T. If this is true for all $t_0$ and $x(t_0)$, the system is said to be completely observable.

Clearly the observability of a system will be a major requirement in filtering and state estimation or reconstruction problems. In many feedback control problems, the controller must use output variable y rather than the state vector x in forming the feedback signals. If the system is observable, then y contains sufficient information about the internal states so that most of the power of state feedback can still be realized.

Besides this, the linear state feedback often cannot be realized directly since it requires that all elements of the state vector be available for measurement. Usually only the output y(t) is known. It is important, therefore, to find an approximation of the state vector knowing only the output and the input of the system. Hence, complete observability will allow construction of good observers.[26]

### 2.3.2 Observability Problem:

A realization {A,B,C} is said to be (state) observable if we can uniquely determine the state x(t) of the realization:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.2}$$

given knowledge of {A,B,C} and { y(t), u(t), t≥0}.

A solution can be obtained in principle from the equations: (see reference [14])

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ \overset{(n-1)}{y}(t) \end{bmatrix} = Ox(t) + T \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ \overset{(n-1)}{u}(t) \end{bmatrix}$$

where

$$O' = \begin{bmatrix} C' & A'C' & . & (A')^{n-1}C' \end{bmatrix} \tag{2.3}$$

and T is the Toeplitz transmission matrix $T = \begin{bmatrix} h_0 & 0 & . & 0 \\ h_1 & . & . & . \\ . & . & . & 0 \\ h_{n-1} & . & h_1 & h_0 \end{bmatrix}$

with $\begin{cases} h_0 = 0 \\ h_i = CA^{i-1}B \end{cases}$ being the Markov parameters.

If the *qnxn* matrix O (Observability matrix) has rank less than n, then some linear combination of the n columns must add to zero, and there are states that will not appear in y(.) or any of its derivatives. Therefore, the summary will be stated in the following theorem:

**Theorem 2.2**

A realization {A,B,C} is state observable if and only if the *qnxn* observability matrix O(C,A) has full rank n.

**proof:** see reference [6-8]

Dual to controllability problem, the following corollary reduces the dimension of the observability matrix.

**Corollary 2.2**

A realization $\{A,B,C\}$ is observable if and only if the $q(n-\bar{q}+1)\times n$ matrix

$$O'_{n-\bar{q}}=\begin{bmatrix}C' & A'C' & \cdots & (A')^{n-\bar{q}}C'\end{bmatrix}$$

where $\bar{q}$ is the rank of C, has full rank n.

**proof:**see reference [6-8]

## 2.4 MINIMAL REALIZATION

With the concepts of controllability and observability , generally the concept of realization is introduced.

Roughly speaking, a realization is the state equation obtained from a transfer function matrix. But the problem of obtaining a transfer function matrix from the state equation will be first introduced.

### 2.4.1 Transfer function matrix from state equations:

For a given state variable mode there is one unique transfer function matrix. The most general state space description of a linear, constant system with p inputs u(t), q outputs y(t), and n state variables x(t) is given by

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)+Eu(t)\end{cases}\tag{2.4}$$

where A, B, C and E are constant matrices of dimensions *nxn*, *nxp*, *qxn*, and *qxp* respectively.

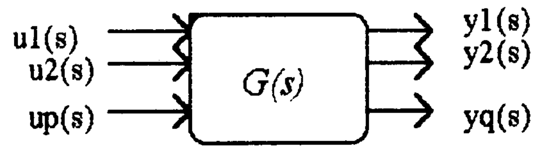


fig.2.1 Block Diagram of a System.

The Laplace transforms of Eq.(2.4) are

$$\begin{aligned} sx(s) - x(t=0) &= Ax(s) + Bu(s) \\ y(s) &= Cx(s) + Eu(s) \end{aligned} \tag{2.5}$$

As usual when dealing with transfer functions, the initial conditions x(t=0) will be ignored. Solving for x(s) gives:

$$x(s) = [sI_n - A]^{-1} Bu(s)$$

Using this result leads to Eq.(2.6), the input-output relationship for the transformed variables:

$$y(s) = \{C[sI_n - A]^{-1} B + E\} u(s) \tag{2.6}$$

The *qxp* matrix which premultiplies u(s) is the ***transfer function matrix*** G(s):

$$G(s) = C[sI_n - A]^{-1} B + E \tag{2.7}$$

A typical element $G_{ij}(s)$ of G(s) is the transfer function relating the ***jth*** input component $u_j$ to the ***ith*** output component $y_i$ $G_{ij}(s) = y_i(s) / u_j(s)$ with all inputs equal to zero except $u_j$ and all initial conditions being zero.

It is convenient to define $\Delta(s) = |sI_n - A|$ then $[sI_n - A]^{-1} = \mathrm{Adj}[sI_n - A] / \Delta(s)$

and the transfer matrix can be rewritten as

$$G(s) = \frac{C\,\mathrm{Adj}[sI_n - A]B + E\Delta(s)}{\Delta(s)} \tag{2.8}$$

Each element of the adjoint matrix is a polynomial in s of degree less than or equal to n-1. Since $\Delta(s)$ is an *n*th degree polynomial in s, each element $G_{ij}(s)$ is a ratio of polynomials in s, with the degree of the denominator at least as great as the degree of the numerator. Such a G matrix is called a ***proper*** rational matrix. If E=[0], then the numerator of every element in G(s) will be of degree less than the denominator. In this case, G(s) is a ***strictly proper*** rational matrix.

Clearly from Eq.(2.8), we have:

$$E = \lim_{s\to\infty} G(s) \tag{2.9}$$

Equation (2.7) indicates that a proper transfer function matrix can be written as the sum of a strictly proper transfer function matrix plus a constant matrix E. When a state variable description of a system is given, the unique corresponding transfer function matrix is given by (2.7). The reverse process is not unique. If the matrix G(s) is given, the matrix E is immediately given by Eq.(2.9). However, the determination of {A,B,C} form a knowledge of G is not a unique process. Many different state variable realizations {A,B,C,E} yield the same transfer function.

If a pair of equations (2.4) can be found which has a specified transfer matrix G(s), then those equations are called ***a realization*** of G(s). For brevity, it is common to refer to the {A,B,C,E} as the realization of G(s). Specifying {A,B,C,E} is equivalent to giving a prescription for synthesizing a system with a given transfer matrix.

### 2.4.2 Definition and implication of irreducible realizations:

If a linear time-invariant dynamical equation is either uncontrollable or unobservable, then there is always a mean to extract the uncontrollable or unobservable part (chapter 4), and result in a controllable or observable realization. Hence there exists a dynamical equation of lesser dimension that has the same transfer function matrix. So this equation is ***reducible***. Otherwise, It will be ***irreducible*** or ***minimal***

**Definition 2.4:**

Of all the possible realizations of G(s), {A,B,C,E} is said to be an ***irreducible (or minimum) realization*** if the associated state space has the smallest possible dimension.

**Important fact:** A minimal realization is both completely controllable and completely observable.[6]

In the case of a scalar transfer function, the minimum dimension required is equal to the order of the denominator of the transfer function after all common pole-zero pair cancellations are made.

Whenever a pole-zero pair is cancelled from a transfer function the system mode associated with the cancelled pole will not be evident in the state equations. Yet, in order to achieve an irreducible realization, these cancellations must be made. What is the implication of irreducible realization in view of this apparent loss of information about the system?

An irreducible realization is a system, of minimal dimension, which is capable of reproducing the measurable relationships between inputs and outputs.

This assumes that the system is originally relaxed, i.e., the initial state vector is zero. For this reason a system and its irreducible realization are said to be zero state equivalent.

Often, the only knowlegde about a system is the information which is obtainable from measurements of inputs and outputs. Transfer functions and transfer function matrices can be experimentally determined from these measurements. The irreducible realization does not cause loss of information in this case, because nothing was known about the system's internal structure in the first place.
If the internal structure of a system is known (for example a circuit diagram) then an irreducible realization may not be the appropriate one to use, state equations can always be written directly from the system's linear graph [3]. The realization {A,B,C,E} obtained in this manner is the most complete system description, whether it is irreducible or not.

In general, transfer matrices, and their irreducible realizations describe only that subsystem which is both completely controllable and completely observable.

## 2.5 HANKEL THEORY

In this section, Markov parameters and Hankel matrices are introduced. They are important in studying various algebraic properties of state-space realizations. Moreover, they have a natural and important interpretation in the discrete time case [14].

### 2.5.1 Single-input single-output case:

In this section, it is dealt with transfer functions.

Consider a proper rational function :

$$\hat{g}(s) = \frac{\beta_0 s^n + \beta_1 s^{n-1} + \ldots + \beta_n}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} \tag{2.10}$$

and expand it into an infinite power series as:

$$\hat{g}(s) = h(0) + h(1)s^{-1} + h(2)s^{-2} + \ldots \tag{2.11}$$

The coefficients {h(i), i=0,1,...} are called the Markov parameters. They can be computed recursively from $\alpha_i$ and $\beta_j$ as

$$
\begin{aligned}
&h(0) = \beta_0 \\
&h(1) = -\alpha_1 h(0) + \beta_1 \\
&. \\
&. \\
&h(n) = -\alpha_1 h(n-1) - \alpha_2 h(n-2) - \ldots - \alpha_n h(0) + \beta_n \\
&h(n+i) = -\alpha_1 h(n+i-1) - \alpha_2 h(n+i-2) - \ldots - \alpha_n h(i), i = 1,2\ldots
\end{aligned}
$$

The Hankel matrix of order $\alpha x \beta$ is defined as :

$$H(\alpha,\beta) = \begin{bmatrix} h(1) & h(2) & . & . & h(\beta) \\ h(2) & h(3) & . & . & h(\beta+1) \\ . & . & & & . \\ . & . & & & . \\ h(\alpha) & h(\alpha+1) & . & . & h(\alpha+\beta-1) \end{bmatrix} \tag{2.12}$$

### 2.5.2 Multiple-input multiple-output case:

In this section, it is dealt with transfer function matrices whose elements are transfer functions .

Consider a $qxp$ proper rational matrix $\hat{G}(s) = \{\hat{g}_{ij}\}$

let $\alpha_i$ :degree of the least common denominator of ith row of $\hat{G}(s)$.

and $\beta_j$: degree of the least common denominator of jth column of $\hat{G}(s)$.

define $\alpha = \max\{\alpha_i, i = 1,2,\ldots,q\}$ and $\beta = \max\{\beta_j, j = 1,2,\ldots,p\}$

The Hankel matrix is formed as:

$$T = \begin{bmatrix} H_{11}(\alpha_1+1,\beta_1) & H_{12}(\alpha_1+1,\beta_2) & \ldots & H_{1p}(\alpha_1+1,\beta_p) \\ H_{21}(\alpha_2+1,\beta_1) & H_{22}(\alpha_2+1,\beta_2) & \ldots & H_{2p}(\alpha_2+1,\beta_p) \\ . & . & \ldots & . \\ H_{q1}(\alpha_q+1,\beta_1) & H_{q2}(\alpha_q+1,\beta_2) & \ldots & H_{qp}(\alpha_q+1,\beta_p) \end{bmatrix} \tag{2.13}$$

where $H_{ij}(\alpha_i+1,\beta_j)$ is the Hankel matrix of $\hat{g}_{ij}(s)$.

Another way is to write the transfer function matrix as:

$$\hat{G}(s) = H(0) + H(1)s^{-1} + H(2)s^{-2} + \ldots$$

where H(i) are $qxp$ constant matrices.

Then the Hankel matrix is formed as:

$$\bar{T} = \begin{bmatrix} H(1) & H(2) & \ldots & H(\beta) \\ H(2) & H(3) & \ldots & H(\beta+1) \\ . & . & \ldots & . \\ H(\alpha+1) & H(\alpha+2) & \ldots & H(\alpha+\beta) \end{bmatrix} \tag{2.14}$$

### 2.5.3 Useful Theorem:

The main problem studied in this thesis is to search for the minimal dimension of a realization, to get the minimal realization. The following theorem gives a direct solution to it.

**Theorem 2.3:**

The proper transfer function $\hat{g}(s)$ has degree n, or the minimal dimension of $\hat{G}(s)$ is n if and only if:

$$\rho H(n,n) = \rho H(n+k, n+l) = n \quad \forall k, l = 1, 2, \ldots$$

**proof:** see reference [6].

This is an interesting result, especially in discrete case, because the pulse-response model $h_T(k)$ may be found on an experimental basis without knowing anything about the underlying system model including its order. Techniques for finding $h_T(k)$ are given in reference[27].

Regardless, though, the Hankel matrix of an arbitrarily high-order NxN may be formed using the $h_T(k)$. Then when the rank of this large Hankel matrix is determined, it will have (in theory) rank n where n is the proper-order minimal state dimension.

# CHAPTER 3

## HANKEL METHODS

There are two generic approaches to find a minimal realization for a given matrix G(s). The first consists of finding a state realization by any available means and then reducing it, if necessary, in the state space domain. This method will be presented in chapter 4.

The second generic approach consists of appropriate reduction of G(s) first and then finding a state variable realization of the reduced transfer function. Some of the methods use Hankel matrices already introduced in chapter 2. Two methods using Hankel matrices will be presented in the following sections. The first one uses the row searching algorithm. The second one uses the singular value decomposition. These two algorithms will also be given.

### 3.1 METHOD 1: **Row searching method**

The following paragraph will deal with the algorithm then the method itself will be presented.

### 3.1.1 Row searching algorithm: [6]

The idea is to search for linear independent rows using elementary row operations. To illustrate the idea , consider the ***nxm*** matrix $A = (a_{ij})$.

*- we check, first, if the first row is a non-zero row.

*-Then, we choose a pivot as a non-zero element in the first row of A, say $a_{1k}$ .

*-we construct the matrix $K_1$ as:

$$K_1 = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ e_{21} & 1 & 0 & . & 0 \\ e_{31} & 0 & 1 & . & 0 \\ . & . & . & . & . \\ e_{n1} & 0 & 0 & . & 1 \end{bmatrix} \qquad (3.1)$$

with $\mathbf{e_{i1} = -a_{ik} / a_{1k}}$ $\mathbf{i = 1, 2, \ldots, n}$. The result is that the *k*th column, except the first element, of $K_1A = (\hat{a}_{ij})$ is a zero column, where $\hat{a}_{ij} = a_{ij} + e_{i1}a_{1j}$.

*-we construct $K_2$, as previously, by choosing a pivot then compute $K_2K_1A$, and so on.

*-If all the elements of an *i*th row are zeros, we let $K_i$ equal the unit matrix.

*-The process is continued until the last row; and we get finally $\overline{A} = K_{n-1}K_{n-2}\ldots K_1A = KA$

*_ Each time we store the *i*th column of $K_i$ in the *i*th column of a matrix F.

The number of non-zero rows in $\overline{A}$ gives the rank of A.

Furthermore the coefficients of combinations of the *j*th linearly dependent row of A, $[b_{j1}, b_{j2}, \ldots, b_{jj}, 0, \ldots, 0]A = 0$ are given by the *j*th row of K, which is computed using the j rows of F following this recursive procedure:

$$b_{jj} = 1$$

$$b_{jk} = \begin{bmatrix} b_{j(k+1)} & b_{j(k+2)} & \cdots & b_{jj} \end{bmatrix} \begin{bmatrix} e_{(k+1)k} \\ e_{(k+2)k} \\ \cdots \\ e_{jk} \end{bmatrix}$$

$$= \sum_{p=k+1}^{j} b_{jp} e_{pk} \qquad k = j-1, j-2, \ldots, 1$$

### 3.1.2 Row searching method:

This method uses of course the row searching algorithm and it is well developed in reference [6].We can distinguish two cases:

#### 3.1.2.1 Single-input single-output case:

From chapter 2 we considered a proper rational function

$$\hat{\mathbf{g}}(\mathbf{s}) = \frac{\boldsymbol{\beta}_0 \mathbf{s}^n + \boldsymbol{\beta}_1 \mathbf{s}^{n-1} + \ldots + \boldsymbol{\beta}_n}{\mathbf{s}^n + \boldsymbol{\alpha}_1 \mathbf{s}^{n-1} + \ldots + \boldsymbol{\alpha}_n} \tag{3.2}$$

it was expanded into an infinite power series of descending power of s as

$$\hat{\mathbf{g}}(\mathbf{s}) = \mathbf{h}(0) + \mathbf{h}(1)\mathbf{s}^{-1} + \mathbf{h}(2)\mathbf{s}^{-2} + \ldots \tag{3.3}$$

The following Hankel matrix is formed:

$$H(n+1,n)=\begin{bmatrix} h(1) & h(2) & \dots & h(n) \\ h(2) & h(3) & \dots & h(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ h(n) & h(n+1) & \dots & h(2n-1) \\ h(n+1) & h(n+2) & \dots & h(2n) \end{bmatrix} \tag{3.4}$$

From this Hankel matrix, the irreducible relization is extracted using the row searching algorithm.

Note that there is one more row than columns, and the Markov parameters up to h(2n) are used in forming H(n+1,n). The row searching algorithm is applied to search the linearly independent rows of H in (3.4) in order from top to bottom.

Let the first $\sigma$ rows be linearly independent and the ($\sigma$+1) row of H be linealy dependent on its previous rows. Then theorem (2.3) implies that the ($\sigma$+k) rows, k=1,2,3..., are all linearly dependent on their previous rows and the rank of H(n+1,n) is $\sigma$. Hence once a linearly dependent row appears in H(n+1,n), the search may be stopped.

The ($\sigma$+1)*th* row of H(n+1,n) is called the *primary* linearly dependent row; the ($\sigma$+k)*th* row, k=2,3,... non-primary linearly dependent rows.

The row searching algorithm will also yield $\{a_i\ ,\ i=1,2,\ldots\}$ such that

$$[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_\sigma\ (1)\ 0 \ldots 0]\mathbf{H}(\mathbf{n}+1,\mathbf{n})=0 \tag{3.5}$$

This equation expresses the primary linearly dependent row as a unique linear combination of its previous rows. The element (1) in equation (3.5) corresponds to the primary dependent row. Note that if $\sigma$=n then $a_i=\alpha_{n-i}$ $i=1,2,\ldots,n$.

It is claimed that the $\sigma$-dimensional dynamical equation:

$$\begin{cases} \dot{x} = Ax + bu \\ y = Cx + eu \end{cases} \tag{3.6}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_\sigma \end{bmatrix} \qquad b = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(\sigma-1) \\ h(\sigma) \end{bmatrix} \tag{3.7}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad e = h(0)$$

is a controllable and observable realization of g(s).

**proof:** see reference [6].

Dual to the introduced procedure, we may also search the linearly independent columns of H(n+1,n) in order from left to right and obtain a different irreducible realization.

### 3.1.2.2 Multiple-input multiple-output case:

we shall search the linear independent rows of a Hankel matrix of G(s) to find an irreducible realization

In Chapter 2 a transfer function matrix G(s) is considered, from which the following Hankel matrix is formed as:

$$T = \begin{bmatrix} H_{11}(\alpha_1+1,\beta_1) & H_{12}(\alpha_1+1,\beta_2) & \ldots & H_{1p}(\alpha_1+1,\beta_p) \\ H_{21}(\alpha_2+1,\beta_1) & H_{22}(\alpha_2+1,\beta_2) & \ldots & H_{2p}(\alpha_2+1,\beta_p) \\ . & . & \ldots & . \\ H_{q1}(\alpha_q+1,\beta_1) & H_{q2}(\alpha_q+1,\beta_2) & \ldots & H_{qp}(\alpha_q+1,\beta_p) \end{bmatrix} \tag{3.8}$$

where $H_{ij}(\alpha_i+1,\beta_j)$ is the Hankel matrix of $\mathbf{g_{ij}(s)}$.

Note that $\left[H_{i1}\, H_{i2} \cdots H_{ip}\right]$, $i = 1, 2, \ldots, q$, have the same number of columns but different number of rows.

Now we search the linearly independent rows of T in order from top to bottom. Because of the structure of T and theorem (2.3) , if one row in $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$ is linearly dependent on its previous rows of T then all subsequent rows of $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$ will also be linearly dependent.

Let $\sigma_i$ be the number of linearly independent rows in $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$. However these linear independent rows may not be all linearly independent in T because they may become dependent on the rows of $\left[H_{j1}\, H_{j2} \ldots H_{jp}\right]$ for $j < i$; hence we have $\sigma_i \le \alpha_i$, $i = 2, 3, \ldots, q$. The row of $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$ which first becomes linearly dependent on its previous rows of T is called the *primary* linearly dependent row of $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$.

Then corresponding to the q primary linearly dependent rows of $\left[H_{i1}\, H_{i2} \cdots H_{ip}\right]$, $i = 1, 2, \ldots, q$ the row searching algorithm will yield

$$
\begin{aligned}
k_1 &= [\overbrace{a_{11}(1)a_{11}(2)\ldots a_{11}(\sigma_1)\ 1}^{\alpha_1+1} \vdots \overbrace{0 \quad \ldots 0 \quad 0\ldots 0 0}^{\alpha_2+1} \vdots \cdots \vdots \overbrace{0 \quad \ldots 0 \quad 0\ldots 0}^{\alpha_q+1}] \\
k_2 &= [a_{21}(1)a_{21}(2)\ldots a_{21}(\sigma_1) 0 \vdots a_{22}(1)\ldots a_{22}(\sigma_2) 1 \ldots 0 0 \vdots \cdots \vdots 0 \quad \ldots 0 \quad 0\ldots 0] \\
&\ \vdots \\
k_q &= [\underbrace{a_{q1}(1)a_{q1}(2)\ldots a_{q1}(\sigma_1)}_{\sigma_1} 0 \vdots \underbrace{a_{q2}(1)\ldots a_{q2}(\sigma_2)}_{\sigma_2} 0 \ldots 0 0 \vdots \cdots \vdots \underbrace{a_{qq}(1)\ldots a_{qq}(\sigma_q)}_{\sigma_q} 1 \ldots 0]
\end{aligned}
\tag{3.9}
$$

such that $k_i T = 0$. The $k_i$ expresses the primary dependent row of $\left[H_{i1}\, H_{i2} \ldots H_{ip}\right]$ as a unique linear combination of its previous linearly independent rows of T.

It is claimed that the following is an irreducible realization of G(s):

$$
\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Eu \end{cases} \tag{3.10}
$$

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix}, \quad E = \begin{bmatrix} h_1(0) \\ h_2(0) \\ \vdots \\ h_q(0) \end{bmatrix} \tag{3.10.a}$$

where for i=1,2,...,q,

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{ii}(1) & -a_{ii}(2) & -a_{ii}(3) & \cdots & -a_{ii}(\sigma_i) \end{bmatrix} \; (\sigma_i \times \sigma_i) \text{ matrix} \tag{3.10.b}$$

for i>j,

$$A_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -a_{ij}(1) & -a_{ij}(2) & \cdots & -a_{ij}(\sigma_j) \end{bmatrix} \; (\sigma_i \times \sigma_j) \text{ matrix} \tag{3.10.c}$$

and, for i=1,2,...,q,

$$b_i = \begin{bmatrix} h_{i1}(1) \\ h_{i1}(2) \\ \vdots \\ h_{i1}(\sigma_i) \end{bmatrix} \tag{3.10.d}$$

The matrix C is given by, for i=1,2,...q, and if $\sigma_i \neq 0$,

$$C = \left.\begin{bmatrix} 1\; 0 \cdots 0 & \vdots & 0\; 0 \cdots 0 & \vdots & & \vdots & 0\; 0 \cdots 0 \\ 0\; 0 \cdots 0 & \vdots & 1\; 0 \cdots 0 & \vdots & & \vdots & 0\; 0 \cdots 0 \\ \vdots\; \vdots \quad \vdots & \vdots & \vdots\; \vdots \quad \vdots & \vdots & \cdots & \vdots & \vdots\; \vdots \quad \vdots \\ \underbrace{0\; 0 \cdots 0}_{\sigma_1} & \vdots & \underbrace{0\; 0 \cdots 0}_{\sigma_2} & \vdots & & \vdots & \underbrace{1\; 0 \cdots 0}_{\sigma_q} \end{bmatrix}\right\} q \tag{3.10.e}$$

If $\sigma_i = 0$ for some i, say i=3 for convenience of illustration, then

$$C = \begin{bmatrix} 1 & \cdots & 0 & \vdots & 0 & \cdots & 0 & \vdots & 00\cdots0 & \vdots & & \vdots & 00\cdots0 \\ 0 & \cdots & 0 & \vdots & 1 & \cdots & 0 & \vdots & 00\cdots0 & \vdots & & \vdots & 00\cdots0 \\ -a_{31}(1) & \cdots & -a_{32}(\sigma_1) & \vdots & -a_{32}(1) & \cdots & -a_{32}(\sigma_2) & \vdots & 00\cdots0 & \vdots & \cdots & \vdots & 00\cdots0 \\ 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 & \vdots & 10\cdots0 & \vdots & & \vdots & 00\cdots0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{\sigma_1} & & & \vdots & \underbrace{0 \quad \cdots \quad 0}_{\sigma_2} & & & \vdots & \underbrace{00\cdots0}_{\sigma_4} & \vdots & & \vdots & \underbrace{10\cdots0}_{\sigma_q} \end{bmatrix} \qquad (3.10.f)$$

**proof:** see reference [6]

Another way is to write the transfer function matrix G(s) as given in chapter 2. Then the Hankel matrix is formed as:

$$\bar{T} = \begin{bmatrix} H(1) & H(2) & \dots & H(\beta) \\ H(2) & H(3) & \dots & H(\beta+1) \\ . & . & \dots & . \\ H(\alpha+1) & H(\alpha+2) & \dots & H(\alpha+\beta) \end{bmatrix} \qquad (3.11)$$

There are $\alpha+1$ block rows in $\bar{T}$; each block has q rows.

The row searching algorithm is applied to search the linearly independent rows of $\bar{T}$ in order from top to bottom. Clearly, if the *i*th row of a block row is linearly dependent on its previous rows, then all the *i*th rows in the subsequent block rows are linearly dependent on their previous rows.

Then the rows of $\bar{T}$ are rearranged into the form:

$$\hat{T} = \begin{bmatrix} H_{11}(\alpha+1,\beta) & H_{12}(\alpha+1,\beta) & \cdots & H_{1p}(\alpha+1,\beta)\} \nu_1 \\ H_{21}(\alpha+1,\beta) & H_{22}(\alpha+1,\beta) & \cdots & H_{2p}(\alpha+1,\beta) \\ \vdots & \vdots & & \vdots \\ H_{q1}(\alpha+1,\beta) & H_{q2}(\alpha+1,\beta) & \cdots & H_{qp}(\alpha+1,\beta)\} \nu_q \end{bmatrix} \tag{3.12}$$

where $\nu_i$ is the number of linearly independent rows of $[H_{i1}\ H_{i2} \cdots H_{ip}]$, $i = 1,2,\ldots,q$. So the primary dependent rows in $\hat{T}$ are the $(\nu_i + 1)$th rows. Clearly the total number of independent rows of $\bar{T}$, $\hat{T}$, and T are all the same. In $\hat{T}$, the linearly independent rows are searched in the order of the first row of $[H_{i1}\ H_{i2} \cdots H_{ip}]$, $i = 1,2,\ldots,q$; the second row of $[H_{i1}\ H_{i2} \cdots H_{ip}]$, $i = 1,2,\ldots,q$, and so forth. For convenience of searching, we search the rows of $\bar{T}$ in order from top to bottom and then rearrange it as $\hat{T}$.

Now from the row searching algorithm and the rearrangements of the coefficients of combinations according to the rearrangements from $\bar{T}$ to $\hat{T}$, the following is obtained:

$$\begin{aligned} \hat{k}_1 &= [\overbrace{a_{11}(1)\cdots a_{11}(\nu_1)10\cdots 0}^{\alpha+1} \vdots \overbrace{a_{12}(1)\cdots a_{12}(\nu_2)00\cdots 0}^{\alpha+1} \vdots\cdots\vdots \overbrace{a_{1q}(1)\cdots a_{1q}(\nu_q)00\cdots 0}^{\alpha+1}] \\ \hat{k}_2 &= [a_{21}(1)\cdots a_{21}(\nu_1)00\cdots 0 \vdots a_{22}(1)\cdots a_{22}(\nu_2)10\cdots 0 \vdots\cdots\vdots a_{2q}(1)\cdots a_{2q}(\nu_q)00\cdots 0] \\ &\ \ \vdots \\ \hat{k}_q &= [\underbrace{a_{q1}(1)\cdots a_{q1}(\nu_1)}_{\nu_1}00\cdots 0 \vdots \underbrace{a_{q2}(1)\cdots a_{q2}(\nu_2)}_{\nu_2}00\cdots 0 \vdots\cdots\vdots \underbrace{a_{qq}(1)\cdots a_{qq}(\nu_q)}_{\nu_q}10\cdots 0] \end{aligned} \tag{3.13}$$

such that $\hat{k}_i\hat{T} = 0$

As previously, it is claimed that the following is also an irreducible realization of G(s):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{3.14}$$

with $$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix} \tag{3.15}$$

where $A_{ij}$, B and C are given as in (3.10) with $\sigma_i$ replaced by $\nu_i$ and $a_{ij}(k)$taken from (3.13).

**proof:** see reference[6]

**Remarks:** _ $\nu_i$, i=1,2,...q are the observability indices of any irreducible realization of G(s).

_ Because of the previous point, we have:

$$\nu = \max\{\nu_i, i = 1,2,\ldots,q\} \le \sigma = \max\{\sigma_i, i = 1,2,\ldots,q\}$$

## 3.2 METHOD 2:Singular value decomposition method

This method uses also a Hankel matrix . It is presented here for comparison purposes for the remaining of the thesis. For more details refer to [6].

### 3. 2.1 Singular value decomposition theorem: [6]

Let H be an ***mxn*** matrix. Then the matrix $H^*H$ has the properties to be a square matrix of order n, it is Hermitian thus its eigenvalues are all real and finally it is positive semidefinite hence its eigenvalues are all nonnegative.

Let $\lambda_i^2$, $i = 1,2,\ldots,n$ be the eigenvalues of $H^*H$, so $\{\lambda_i \ge 0,\ i = 1,2,\ldots,n\}$is the set of *singular values* of H.

For convenience we arrange the set such that $\lambda^2_1 \ge \lambda^2_2 \cdots \ge \lambda^2_n \ge 0$.

If the rank of H , thus of $H^*H$, is r, then the last (n-r) eigenvalues are equal to zero. Let $Q = [q_1\, q_2 \cdots q_r \vdots q_{r+1} \cdots q_n] = [Q_1 \vdots Q_2]$ be a matrix whose columns are the orthonormal eigenvectors of $H^*H$. Then the following theorem will be used:

**Theorem 3.1:**

A Hermitian matrix M can be transformed by a unitary matrix into a diagonal matrix with real elements; or equivalently, for any hermitian matrix M there exists a nonsingular matrix P with the property $P^{-1} = P^*$ such that $\hat{M} = PMP^*$ where $\hat{M}$ is a diagonal matrix with the real eigenvalues of M on the diagonal.

**proof:** see reference[6]

This theorem implies that

$$Q^*H^*HQ = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.16}$$

where $\Sigma^2 = \text{diag}\{\lambda_1{}^2, \lambda^2{}_2, \ldots, \lambda_r^2\}$.

Equation (3.16) will become

$$\begin{cases} Q_2^*H^*HQ_2 = 0 \\ Q_1^*H^*HQ_1 = \Sigma^2 \end{cases} \tag{3.17}$$

which implies

$$\Sigma^{-1} Q_1^*H^*HQ_1 \Sigma^{-1} = I \tag{3.18}$$

where $\Sigma = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$.

Let $R_1$ an mxn matrix defined as: $R_1 \equiv HQ_1 \Sigma^{-1}$ and $R_2$ chosen so that $R = [R_1\ R_2]$ is unitary. R will be orthonormal. The result of all this will be:

$$R^*HQ = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{3.19}$$

This is summarized in a theorem called *singular value decomposition theorem.*

**Theorem 3.2**

Every $m x n$ matrix H of rank r can be transformed into the form:

$$R^{*}HQ = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad H = R\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}Q^{*} \tag{3.20}$$

where $R^{*}R = RR^{*} = I_m$, $Q^{*}Q = QQ^{*} = I_n$, and $\Sigma = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$.

**proof:**see reference [6].

### 3.2.2 Singular value decomposition method:

Consider the $q x p$ proper rational matrix G(s) given as in chapter 2.

Let $\Psi(s) = s^m + \alpha_1 s^{m-1} + \cdots + \alpha_m$ be the least common denominator of all elements of G(s).

The following Hankel matrices are defined as:

$$T = \begin{bmatrix} H(1) & H(2) & \cdots & H(m) \\ H(2) & H(3) & \cdots & H(m+1) \\ \vdots & \vdots & & \vdots \\ H(m) & H(m+1) & \cdots & H(2m-1) \end{bmatrix} \tag{3.21}$$

and

$$\tilde{T} = \begin{bmatrix} H(2) & H(3) & \cdots & H(m+1) \\ H(3) & H(4) & \cdots & H(m+2) \\ \vdots & \vdots & & \vdots \\ H(m+1) & H(m+2) & \cdots & H(2m) \end{bmatrix} \tag{3.22}$$

The singular value theorem (theorem3.2) is used to find an irreducible realization directly from these Hankel matrices.

This theorem implies the existence of a *qmxqm* and *pmxpm* unitary matrices K and L such that:

$$T = K\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}L \tag{3.23}$$

where $\Sigma = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\lambda_i, i = 1, 2, \ldots, n$ are the positive square roots of the eigenvalues of $T^*T$ where n is the rank of T.

Let $K_1$ and $L_1$ denote respectively the first n columns of K and the first n rows of L. Then T will be written as

$$T = K_1 \Sigma L_1 = K_1 \Sigma^{1/2} \Sigma^{1/2} L_1 \equiv VU \tag{3.24}$$

where _ $\Sigma^{1/2} = \text{diag}\{\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\}$.

_ $V \equiv K_1 \Sigma^{1/2}$ a *qmxn* matrix.

_ $U \equiv \Sigma^{1/2} L_1$ a *nxpm* matrix.

Let the pseudoinverses of V and U be defined as:

$$V^+ = \Sigma^{-1/2} K_1^* \quad \text{and} \quad U^+ = L_1^* \Sigma^{-1/2} \tag{3.25}$$

The irreducible realization is defined in the following theorem:

**Theorem 3.3:**

Considering all the parameters defined previously and using the singular value decomposition, the realization {A,B,C,E} defined by:

$$A = V^{+}\tilde{T}U^{+} \tag{3.26.a}$$

$$B = \text{first p columns of } U = UI'_{p,pm} \tag{3.26.b}$$

$$C = \text{first q rows of } V = I_{q,qm}V \tag{3.26.c}$$

$$E = H(0)$$

is an irreducible realization of G(s).

**proof:** see reference[6]

In summary, these two methods use a Hankel matrix in starting the search. A comparison between them and a third modifed Chen's method will be presented in chapter 7.

CHAPTER 4

# KALMAN METHODS

Kalman methods use the Kalman decomposition theorem. It is an important conceptual tool because it clarifies a number of concepts and problems.

Consider the dynamical equation :

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Eu \end{cases} \tag{4.1}$$

where A,B,C, and E are ***nxn***, ***nxp***, *qxn*, *qxp* real constant matrices. For simplicity we take E=0. (If E is different from zero then it will be clear that its contribution can be added at the end).

The intuition is the following: we separate the unobservable part and uncontrollable part both at the same time. We expect four interacting subsystems: the controllable observable system, the controllable unobservable system, the uncontrollable observable system and of course the uncontrollable unobservable system, which will be labeled using obvious notations $c.o$, $c.\not{o}$, $\not{c}.o$, $\not{c}.\not{o}$.

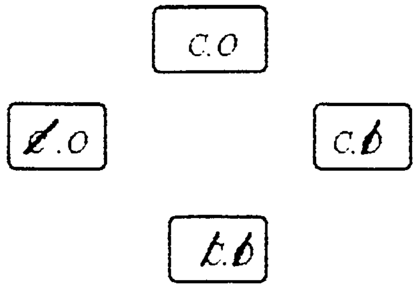

fig 4.1 Block Diagram of the Kalman decomposition

In the following sections a brief review on similarity transformations will be given, then the controllability and observability decomposition will be stated, and finally the general Kalman decomposition theorem will be presented.

### 4.1 SIMILARITY TRANSFORMATIONS

Two realizations are similar to each other (or related by a similarity transformation) if there exists a nonsingular matrix T such that

$$\overline{A} = T^{-1}AT\ , \qquad \overline{B} = T^{-1}B\ , \qquad \overline{C} = CT \tag{4.2}$$

It can be seen that the states of two realizations can be related as: $x(t) = T\,\overline{x}(t)$ so that equation (4.1) will become:

$$\begin{cases} \dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u \\ y = \overline{C}\overline{x} + \overline{E}u \end{cases} \tag{4.3}$$

but that the transfer functions are unaffected,

$$G(s) = C(sI - A)^{-1}B = \overline{C}(sI - \overline{A})^{-1}\overline{B} \tag{4.4}$$

Furthermore, we have

$$\bar{O} = OT\,, \quad \bar{C} = T^{-1}C \tag{4.5}$$

So that the ranks of the observability and controllability matrices are preserved under similarity transformations and this leads to the following theorem.

**Theorem 4.1**

The controllability and observability of a linear time-invariant dynamical equation are invariant under an equivalence (similarity) transformation.

**proof:** see reference[6]

The dynamical equations (4.1) and (4.3) are said to be ***equivalent***.

It is therefore natural to look for similarity transformations that will give realizations in special forms (minimal) that may be especially suited to particular applications.

## 4.2 CONTROLLABILITY DECOMPOSITION

Because the controllability of a linear dynamical equation is invariant under a similarity transformation, the following theorem may be stated.

**Theorem 4.2: Displaying the controllable and non-controllable states**

Suppose {A,B,C} is a realization such that the controllability matrix Cn(A,B) given in (2.1) has rank r less than or equal to n.

Kalman has shown that we can find a similarity transformation (matrix) T such that $\bar{A}, \bar{B}$ and $\bar{C}$ in (4.2) have the following forms:

$$\overline{A} = \begin{bmatrix} \overline{A}_c & \vdots & \overline{A}_{12} \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \overline{A}_{\bar{c}} \end{bmatrix} \begin{matrix} r \\ . \\ n-r \end{matrix}, \overline{B} = \begin{bmatrix} \overline{B}_c \\ \cdots \\ 0 \end{bmatrix} \begin{matrix} r \\ . \\ n-r \end{matrix}, \overline{C} = \begin{bmatrix} \overline{C}_c & \vdots & \overline{C}_{\bar{c}} \end{bmatrix} \tag{4.6}$$

where: 1) $(\overline{A}_c, \overline{B}_c)$ is controllable.

2)the states of $\{\overline{A}, \overline{B}, \overline{C}\}$ can be clearly separated into controllable and non-controllable states.

3) $\overline{C}_c(sI - \overline{A}_c)^{-1}\overline{B}_c = C(sI - A)^{-1}B$

Since Cn(A,B) has rank r, we can find r linearly independent columns, say $\{f_1, \ldots, f_r\}$, that span the np columns of Cn(A,B). Let $\{\theta_{r+1}, \ldots, \theta_n\}$ be n-r columns such that $T \equiv [f_1, \ldots, f_r \theta_{r+1}, \ldots, \theta_n]$ has full rank n. Then T is the appropriate similarity transformation to the system.

**proof:** see reference [14]

### 4.3 OBSERVABILITY DECOMPOSITION

Similarly, we can find a choice of state variables in which observable and non-observable states can be clearly identified.

**Theorem 4.3: Displaying observable and non-observable states**

Let {A,B,C} be a realization such that the observability matrix O(C,A) given in (2.3) has a rank r less than or equal to n. Kalman has shown that we can find a similarity transformation (matrix) T such that $\overline{A}, \overline{B}$ and $\overline{C}$ in (4.2) have the following forms:

$$\bar{A} = \begin{array}{c} \begin{bmatrix} \bar{A}_o & \vdots & 0 \\ \cdots & \vdots & \cdots \\ \underset{r}{\bar{A}_{21}} & \vdots & \underset{n-r}{\bar{A}_{\bar{o}}} \end{bmatrix} \end{array} \begin{array}{l} r \\ . \\ n-r \end{array}, \bar{B} = \begin{bmatrix} \bar{B}_o \\ \cdots \\ \bar{B}_{\bar{o}} \end{bmatrix}, \bar{C} = \begin{bmatrix} \bar{C}_o & \vdots & 0 \end{bmatrix} \tag{4.7}$$

where: 1)_ $(\bar{A}_o, \bar{C}_o)$ is observable.

2)_ The states of $\{\bar{A}, \bar{B}, \bar{C}\}$ can be clearly separated into observable and non_observable states

3)_ $\bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o = C(sI - A)^{-1}B$

Since O(C,A) has rank r, we can find r linearly independent rows, say $\{f_1, \ldots, f_r\}'$, that span the qn rows of O(C,A). Let $\{\theta_{r+1}, \ldots, \theta_n\}'$ be n-r rows such that $T \equiv [f_1, \ldots, f_r \theta_{r+1}, \ldots, \theta_n]'$ has full rank, then T is the appropriate similarity transformation.

**proof:**see reference [14]

## 4.4 KALMAN DECOMPOSITION

The above results can be put together to obtain a decomposition of any realization into four parts that clearly have the properties of joint controllability and observability, controllability but not observability, observability but not controllability, and neither controllability nor observability. Combining the two previous theorems, we have the following.

### Theorem 4.4: canonical decomposition theorem

If we consider the linear time-invariant equation (4.1) , by equivalence transformation it can be transformed to the following canonical form:

$$\begin{cases}\begin{bmatrix}\dot{\bar{x}}_{c\bar{o}}\\ \dot{\bar{x}}_{co}\\ \dot{\bar{x}}_{\bar{c}}\end{bmatrix}=\begin{bmatrix}\bar{A}_{c\bar{o}} & \bar{A}_{12} & \vdots & \bar{A}_{13}\\ 0 & \bar{A}_{co} & \vdots & \bar{A}_{23}\\ \cdots & \cdots & \vdots & \cdots\\ 0 & 0 & \vdots & \bar{A}_{\bar{c}}\end{bmatrix}\begin{bmatrix}\bar{x}_{c\bar{o}}\\ \bar{x}_{co}\\ \bar{x}_{\bar{c}}\end{bmatrix}+\begin{bmatrix}\bar{B}_{c\bar{o}}\\ \bar{B}_{co}\\ 0\end{bmatrix}u\\ y=\begin{bmatrix}0 & \bar{C}_{co} & \vdots & \bar{C}_{\bar{c}}\end{bmatrix}\bar{x}+Eu\end{cases} \tag{4.8}$$

where the vector $\bar{x}_{c\bar{o}}$ is controllable but not observable, $\bar{x}_{co}$ is controllable and observable, and $\bar{x}_{\bar{c}}$ is not controllable. Furthermore, the transfer function of (4.1) is:

$$C_{co}(sI-\bar{A}_{co})^{-1}\bar{B}_{co}+E \tag{4.9}$$

which depends only on the controllable and observable part of the dynamical equation(4.1)

**proof**: see reference [6]

An important property of minimal realizations is that they can all be related to one another by a unique similarity transformation.

**Theorem 4.5:**

If $\{A_i, B_i, C_i,\ i=1,2\}$ are two minimal realizations of a transfer function, there exists a unique invertible matrix T such that

$$A_2=T^{-1}A_1T\,,\quad B_2=T^{-1}B_1\,,\quad C_2=C_1T \tag{4.10}$$

Furthermore, T can be specified as:

$$\begin{aligned}T&=Cn_1Cn_2'(Cn_2Cn_2')^{-1}\\ T^{-1}&=(O_2'O_2)^{-1}O_2'O_1\end{aligned} \tag{4.11}$$

**proof:** see reference [14].

# CHAPTER 5

# PROPOSED METHODS

The contribution of this thesis consists of the elaboration of a recursive algorithm to search the linear independent vectors in a set. This algorithm is based on the linear independence of orthogonally projected vectors. It is efficient and numerically stable. It will be used in improving the row searching method proposed by Chen and Mital [6-8] (see chapter 3), and Kalman method proposed by Kalman [15-17] (see chapter 4).

So, in the next section, the recursive algorithm will be stated, then the modified methods will be presented.

## 5.1 RECURSIVE ALGORITHM

This algorithm searches sequentially in a set of vectors $\{V_1, V_2, \ldots, V_n\}$ to determine whether or not the vector $V_k$ is linearly independent of the previous ones. It first checks whether or not the first vector is non-zero, or equivalently, linearly independent by itself, then it checks whether or not the second vector is linearly independent of the first one and so on.

**ALGORITHM:**

Given a set of s-dimensional vectors $\{V_1, V_2, \ldots, V_n\}$, a $s$x$s$ matrix P(k) is determined recursively for k=1,2,...,n.

1)_Initialise $P(0) = I_s$ [Identity matrix].

2)_For k=1,2,...,n do

_if $V_k^T P(k-1)V_k \neq 0$, then

$$P(k) = P(k-1) - \frac{[P(k-1)V_k][P(k-1)V_k]^T}{V_k^T P(k-1)V_k} \quad (5.1)$$

and $V_k$ is linearly independent of the rest of the set.

_else P(k)=P(k-1) and the vector is linearly dependent.

**REMARK:** The matrix P is a projection matrix since it satisfies:

_P is symmetric.

_$P^2 = P$ (idempotent).

**PROOF OF THE ALGORITHM:** we'll use recursion.

i) for i=1 $P(0)V_1 \neq 0 \Rightarrow V_1 \neq 0$ and $P(1) = P(0) - \frac{[P(0)V_1][P(0)V_1]^T}{V_1^T P(0)V_1} = I_n - \frac{V_1 V_1^T}{V_1^T V_1}$

for i=2, taking $V_2$ linearly independent of $V_1$,we must have $V_2^T P(1)V_2 \neq 0$ or simply verify that

$$\begin{cases} P(1)V_1 = 0 \\ P(1)V_2 \neq 0 \end{cases} \quad (5.2)$$

we'll check if this is so:

_$P(1)V_1 = I_n V_1 - \frac{V_1 V_1^T}{V_1^T V_1} V_1 = V_1 - V_1 \frac{V_1^T V_1}{V_1^T V_1} = 0$

$$P(1)V_2 = V_2 - V_1 \frac{V_1^T V_2}{V_1^T V_1} \neq 0 \quad (V_2 \text{ linearly independent of } V_1)$$

now suppose that $V_2 = \alpha V_1$ (they are linearly dependent) then

$$P(1)V_2 = \alpha V_1 - \alpha V_1 \frac{V_1^T V_1}{V_1^T V_1} = 0 \qquad \text{so} \qquad V_2^T P(1)V_2 = 0$$

ii)we suppose it is true for k; that is, if $V_k$ is linearly independent of the set $\{V_1,\dots,V_{k-1}\}$, then

$$\begin{cases} P(k-1)V_i = 0, i < k \\ P(k-1)V_k \neq 0 \end{cases} \tag{5.3}$$

we must verify the statement for k+1 or: if $V_{k+1}$ is linearly independent of the set $\{V_1,\dots,V_k\}$ then:

$$\begin{cases} P(k)V_i = 0, i < k+1 \\ P(k)V_{k+1} \neq 0 \end{cases} \tag{5.4}$$

<u>i<k:</u> $$P(k)V_i = \begin{aligned} & P(k-1)V_i - \frac{[P(k-1)V_k][P(k-1)V_k]^T}{V_k^T P(k-1)V_k} V_i \\ & = 0 - \frac{P(k-1)V_k V_k^T}{V_k^T P(k-1)V_k} P(k-1)V_i = 0 \end{aligned}$$

<u>i=k:</u> $$P(k)V_k = P(k-1)V_k - P(k-1)V_k \frac{V_k^T P(k-1)V_k}{V_k^T P(k-1)V_k} = 0$$

<u>i=k+1:</u> $$P(k)V_{k+1} = P(k-1)V_{k+1} - P(k-1)V_k \frac{V_k^T P(k-1)V_{k+1}}{V_k^T P(k-1)V_k} \neq 0$$

$$\Rightarrow V_{k+1}^T P(k) V_{k+1} \neq 0$$

so we have proved the first point. Now we suppose that $V_{k+1}$ is linearly dependent of the set, say $V_{k+1} = \alpha_k V_k + \alpha_{k-1} V_{k-1} + \ldots + \alpha_1 V_1$ then

$$P(k)V_{k+1} = P(k-1)V_{k+1} - \frac{P(k-1)V_k V_k^T}{V_k^T P(k-1)V_k} P(k-1)V_{k+1}$$

$$P(k)V_{k+1} = P(k-1)\left[\alpha_k V_k + \sum_{i=1}^{k-1} \alpha_i V_i\right] - \frac{P(k-1)V_k V_k^T}{V_k^T P(k-1)V_k} P(k-1)\left[\alpha_k V_k + \sum_{i=1}^{k-1} \alpha_i V_i\right]$$

$$P(k)V_{k+1} = P(k-1)\alpha_k V_k - \frac{P(k-1)V_k V_k^T}{V_k^T P(k-1)V_k} P(k-1)\alpha_k V_k$$

$$P(k)V_{k+1} = \alpha_k P(k-1)V_k - \alpha_k P(k-1)V_k \frac{V_k^T P(k-1)V_k}{V_k^T P(k-1)V_k} = 0$$

$$\Rightarrow \mathrm{V}_{k+1}^T \mathrm{P}(\mathrm{k}) \mathrm{V}_{k+1} = 0$$

So we proved that whether $V_k^T P(k-1)V_k$ is equal to zero or not, we can tell if $V_k$ is linearly independent of the previous vectors of the set or not

## 5.2 MODIFIED CHEN's METHOD

Chen and Mital [6-8] proposed a method based on the Hankel matrix and consists of searching its linear independent rows, and obtaining a minimal realization from the coefficients of linear combinations corresponding to the primary linearly dependent rows of the Hankel matrix (chapter 3).

In the method proposed in this thesis, the search will be done using the recursive algorithm presented in section (5.1).

The method consists of:

**1.Construction of the Hankel matrix:**

It is as presented by Chen and Mital.see chapter 3, section 3.1.2..

**2.Search of the rows of the Hankel matrix:**

The recursive algorithm is used as presented (previously).

The coefficients of combination of the jth linearly dependent row on its previous (j-1) rows are computed by solving a standard equation of the type Ax=b.

**3.Construction of minimal realizations:**

Once we get which rows of the Hankel matrix are primary linearly dependent of the previous ones, and computed the coefficients of combinations, we construct the minimal realization as given in Chen and Mital's method. (see chapter 3, section3.1.2.)

## 5.3 MODIFIED KALMAN METHOD

Kalman gave a solution to the problem of obtaining a minimal realization which is based on similarity transformation determined from the controllability and observability matrices. Our method uses the same similarity transformation but starting from a controllable or observable realization, which is easily obtained. The linear independence of the rows (columns) of the observability (controllability) matrix will be searched using the recursive algorithm.

### 5.31 Reduction of a controllable realization:

The proposed method consists in constructing directly a controllable realization, then using a similarity transformation to reduce it to a minimal one.

**a.Construction of a controllable realization:**

Let $G(s) = \hat{G}(s) + G(\infty)$ so that $\hat{G}(s)$ is strictly proper, and $\Psi(s)$, its monic least common denominator of the form:

$$\Psi(s) = s^m + \alpha_1 s^{m-1} + \ldots + \alpha_m \tag{5.5}$$

We can write:

$$\hat{G}(s) = \frac{1}{\Psi(s)}\left[R_1 s^{m-1} + R_2 s^{m-2} + \ldots + R_m\right] \tag{5.6}$$

where $R_i$ are $q$x$p$ constant matrices. The following dynamical equation:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0_p & I_p & 0_p & \cdots & 0_p \\ 0_p & 0_p & I_p & \cdots & 0_p \\ . & . & . & \cdots & . \\ 0_p & 0_p & 0_p & \cdots & I_p \\ -\alpha_m I_p & -\alpha_{m-1} I_p & -\alpha_{m-2} I_p & \cdots & -\alpha_1 I_p \end{bmatrix} x + \begin{bmatrix} 0_p \\ 0_p \\ . \\ 0_p \\ I_p \end{bmatrix} u \\ y = \begin{bmatrix} R_m & R_{m-1} & R_{m-2} & \cdots & R_1 \end{bmatrix} x + G(\infty) u \end{cases} \tag{5.7}$$

is a controllable realization of $\hat{G}(s)$.

**proof**: see reference [6]

**2.Reduction Method:**

As suggested by Kalman, we can find a similarity transformation matrix to reduce the controllable realization (5.7) to a minimal one. The algorithm proposed here is outlined as follows:

1)_Construct the observability matrix O(C,A)

2)_Search for the r linear independent rows of O(C,A), using the recursive algorithm.

3)_Construct the (n-r) linear independent rows to construct a non-singular similarity transformation T.

4)_Apply the transformation to the controllable realization to get a new realization in the form of equation (4.7) in chapter 4.

5)_Extract the new controllable and observable, hence minimal, realization.

We'll develop each point in detail:

1)-Construction of the observability matrix O(C,A):

Using the controllable realization obtained in (5.7), compute the observability matrix as:

$$O(C,A)=\begin{bmatrix}C' & A'C' & \dots & (A')^{n-1}C'\end{bmatrix}' \tag{5.8}$$

or, if $\bar{q}$ is the rank of C then

$$O(C,A)=\begin{bmatrix}C' \ \mathrm{A'C'} \dots (\mathrm{A'})^{n-\bar{q}}\mathrm{C'}\end{bmatrix}' \tag{5.8.a}$$

Because the initial matrix may be block observable and if r=n/q is an integer then the observability matrix is:

$$O(C,A)=\begin{bmatrix}C' & A'C' & \dots & (A')^{r-1}C'\end{bmatrix}' \tag{5.8.b}$$

2)-Searching the linear independent rows of O(C,A):

The recursive algorithm is used.

3)-New linear independent rows computation: In order to construct the similarity transformation T we do the following:

let $V_k$ be a vector linearly dependent of the set $\{V_1,\dots,V_{k-1}\}$ then we can write:

$$V_k=\alpha_1 V_1+\dots+\alpha_{k-1}V_{k-1},\ \text{or}\begin{pmatrix}v_{k1}\\ v_{k2}\\ \vdots\\ v_{kn}\end{pmatrix}=\begin{pmatrix}\alpha_1 v_{11}+\dots+\alpha_{k-1}v_{k-1\,1}\\ \alpha_1 v_{12}+\dots+\alpha_{k-1}v_{k-1\,2}\\ \vdots\\ \alpha_1 v_{1n}+\dots+\alpha_{k-1}v_{k-1\,n}\end{pmatrix} \tag{5.9}$$

let a new vector $\qquad nV_k=\mathrm{diag}(v_k)\times V_k \qquad (5.10)$

where $\mathrm{diag}(V_k)$ is a square matrix whose main diagonal elements are the elements of the vector $V_k$.

$$\text{Then } nV_k = \begin{pmatrix} v_{k1}^2 \\ \vdots \\ v_{kn}^2 \end{pmatrix} = \begin{pmatrix} (\alpha_1 v_{11} + \ldots + \alpha_n v_{k-1\,1})^2 \\ \vdots \\ (\alpha_1 v_{1n} + \ldots + \alpha_n v_{k-1\,n})^2 \end{pmatrix} \tag{5.11.a}$$

$$nV_k = \begin{pmatrix} \alpha_1 v_{11}^2 + \ldots + \alpha_n v_{k-1\,1}^2 + \alpha_1\alpha_2 v_{11} v_{12} \cdots \\ \vdots \\ \alpha_1 v_{1n}^2 + \ldots + \alpha_n v_{k-1\,n}^2 + \alpha_1\alpha_2 v_{1n} v_{2n} \cdots \end{pmatrix} \tag{5.11.b}$$

the last equation shows clearly that the new vector is linearly independent of the set.

Remark: In case the elements of the linearly dependent vector $V_k$ are all ones or/and zeroes then additionning a random number to any of its elements will be sufficient to get a linearly independent vector.

4)_We apply the similarity transformation to the controllable realization obtained in (5.7) to get a new realization in the form of equation (4.7) in chapter 4.

5)_ We extract the minimal realization already separated by the previous computations.

**Remark**: We will obtain for the new C: $\overline{C}_o = \begin{bmatrix} I_q & \vdots & 0 & \ldots & 0 \end{bmatrix}$

### 5.3.2 Reduction of an observable realization:

It consists first in constructing an observable realization, then applying an algorithm to perform a reduction in order to obtain a minimal realization.

**1.Construction of an observable realization:**

Let the transfer function matrix be written as:

$$G(s) = H(0) + H(1)s^{-1} + H(2)s^{-2} + \ldots \tag{5.12}$$

where H(i) are the *qxp* constant matrices constructed from the Markov parameters. Let the monic least common denominator be given by equation (5.5).

If $\{A,B,C,E\}$ is a realization of the transfer function then we can write:

$$G(s) = E + C(sI - A)^{-1}B = E + CBs^{-1} + CABs^{-2} + \ldots$$

and by coefficient identification we have:

$$\begin{cases} E = H(0) \\ H(i+1) = CA^{i}B, \ i = 0,1,2 \end{cases}$$

The qm_dimensional observable realization is then:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0_q & I_q & \cdots & 0_q \\ 0_q & 0_q & \cdots & 0_q \\ \vdots & \vdots & \cdots & \vdots \\ 0_q & 0_q & \cdots & I_q \\ -\alpha_m I_q & -\alpha_{m-1} I_q & \cdots & -\alpha_1 I_q \end{bmatrix} x + \begin{bmatrix} H(1) \\ H(2) \\ \vdots \\ H(m-1) \\ H(m) \end{bmatrix} u \\ y = \begin{bmatrix} I_q & 0_q & \cdots & 0_q \end{bmatrix} x + H(0)u \end{cases} \tag{5.13}$$

**proof:** see reference [6]

## 2. Reduction method:

As suggested by Kalman, we can find a similarity transformation to extract the controllable part of the already observable realization, thus obtaining a minimal one.

The steps of the proposed algorithm are:

1_ Write the transfer function as in (5.12), in order to get the observable realization.

2_ Construct its controllability matrix Cn(A,B).

3_ Search for the r linear independent columns of Cn(A,B), using a recursive algorithm.

4_ Construct the new (n-r) linear independent columns so that the similarity transformation T will be non_singular.

5_ Compute equation (4.2) to get a new realization in the form of (4.6) in chapter4.

6_ Extract the new controllable and observable, hence minimal, realization.

Each point is developed in detail.

1_ To write the transfer function as in (5.12), we have only to compute the Markov parameters of each of its elements.

2_ To construct the controllability matrix Cn(A,B): We use the observable realization obtained in (5.13), and Cn(A,B) is constructed as follows:

$$\mathbf{Cn(A,B)} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \ldots & \mathbf{A^{n-1}B} \end{bmatrix} \qquad (5.14)$$

or if $\bar{p}$ is the rank of B then

$$\mathrm{Cn(A,B)} = \begin{bmatrix} \mathrm{B} & \mathrm{AB} & \ldots & \mathrm{A}^{n-\bar{p}}\mathrm{B} \end{bmatrix} \qquad (5.14.a)$$

If our initial matrix is a block controllable matrix and r=n/p is an integer we can construct the observability matrix as:

$$\mathbf{Cn(A,B)} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A^{r-1}B} \end{bmatrix} \qquad (5.14.b)$$

3_ To search the linear independent columns of Cn(A,B) the recursive algorithm is used.

4_ To construct the similarity transformation T the procedure presented in section 5.3.1 will be used.

5_ Perform the computations as in equation (4.2) in chapter 4 to get a new realization.

6_ Extract the new irreducible realization .

**Remark:** $\overline{B}_c$ will be obtained in the form of: $\overline{B}_c = \left[I_p, 0 \ldots 0\right]'$

**SUMMARY:**

This chapter deals essentially with our contribution in this field. The most important thing here is the recursive algorithm which can be used in many areas where linear independence is involved. The innovation in the modified methods presented here is of course the use of this algorithm and, for the case of the modified Kalman decomposition, the fact that the reduction is done only once.

The next chapter will deal with examples which will illustrate each method presented here. A comparison between them will also be presented.

# CHAPTER 6

# IMPLEMENTATION AND NUMERICAL RESULTS

In order to test the algorithms described in the previous chapters we have developed a set of computer programs. One of the best computer tools to handle matrices is the software package MATLAB [21].

## 6.1 MATLAB FEATURES

MATLAB is an interactive program used in scientific and engineering numeric calculations. It is an interactive system whose basic data element is a matrix that does not require dimensioning. Furthermore, problem solutions are expressed in MATLAB almost exactly as they are written mathemetically.

MATLAB works with essentially one kind of object, a rectangular matrix with possibly complex elements. In some situations, special meaning is attached to 1-by-1 matrices, which are scalars, and to matrices with only one row or column, which are vectors. Operations and commands in MATLAB are intended to be natural, in a matrix sense, not unlike how they might be indicated on paper.

The main features used in the following algorithms range from the simplest matrix operations, and array operations, to matrix functions such as singular value decomposition, rank, controllability and observability matrices, etc...

Kalman decomposition is also available in MATLAB which allowed a comparison between the results obtained using the algorithm and the results given directly by this function.

## 6.2 ALGORITHMS

First, the old algorithms will be stated as used then the new algorithms will be given, in order to compare their efficiency. In the following, we consider a system described by a *qxp* transfer function matrix, the degree of its monic least common denominator is m, and $\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$are as defined in chapter3.

### 6.2.1 Chen's algorithm:

It consists of first constructing the Hankel matrix of dimension $((\boldsymbol{\alpha}+1)\times q)\times(\boldsymbol{\beta}\times p)$, then to search this Hankel matrix using the row searching algorithm . After that, the coefficients of linear combination of the primary dependent rows are computed (see chapter3 method1).

cost: For a Hankel matrix of order *nxr*, the algorithm needs $\frac{n(n-1)}{2}+n^2r(n-2)$flops to construct the two matrices K and F of dimension *nxn*.

To compute the coefficients, supposing that the last row is dependent, it needs $\frac{n(n-1)}{2}$ flops.

### 6.2.2 Singular value decomposition algorithm:

First, two Hankel matrices are constructed, then a singular value decomposition is applied to one of them using the available function in MATLAB. Thus obtaining Three matrices from which the irreducible realization is extracted (see chapter3 method2).

**cost**:Space must be available for the two Hankel matrices and The three matrices obtained after the decomposition.The algorithm consists of multiplication of matrices. Inversion and pseudo-inversion of matrices are also involved. The computations are of order $\mathbf{O(n^3)}$.

### 6.2.3 Kalman decomposition:

For this case, we used directly the function available in MATLAB.

### 6.2.4 Recursive algorithm:

The recursive algorithm as introduced in the previous chapters searches for linear independent vectors in a set using orthogonally projected vectors. It consists of computing each time a matrix P and a value t equal to the multiplication of the transpose of the reached vector by P by the vector.

**cost**: For a set of m n-dimensional vectors it will use a matrix of dimension $n$x$n$ and it needs $\mathbf{m(3n^2+n(n+1))}$ flops.

### 6.2.5 Modified Chen's algorithm:

The modified Chen's algorithm is slightly different from the Chen's row searching algorithm. It starts with a Hankel matrix of dimension $((\boldsymbol{\alpha}+1)\times q)\times(\boldsymbol{\beta}\times p)$ then uses the recursive algorithm to search for its linear independent rows. Then, for our case, the improved Gauss-Jordan algorithm is used to compute the coefficients of combination of the primary linearly dependent rows. The construction of the irreducible realization is straightforward as described in chapter3 section 3.1.2..

**cost:** Let the Hankel matrix has a dimension of *n*x*r*, then the recursive algorithm needs $\mathbf{n(r(r+1)+3r^2)}$ flops and space for a single matrix of dimension *n*x*r*. The improved Gauss-Jordan algorithm uses $\mathbf{\frac{1}{3}n^3+n^2}$ flops.

### 6.2.6 Controllable realization reduction algorithm:

This algorithm starts by an already controllable realization of dimension *mp*x*mp*. Then the observability matrix is constructed and searched in order to find the transformation matrix to reduce it to a minimal one(see chapter5 section5.3.1).

**cost:** Space is needed for the controllable realization {A,B,C} of dimension *mp*x*mp*, *mp*x*p* and *q*x*mp* respectively. Then the observability matrix is computed which needs *q*(*m*-1)x*mp* space and $\mathbf{((mp)^2q)(m-2)}$flops. The recursive algorithm is used to search this matrix then it is transformed to the similarity matrix and its inverse needs the same amount of space and $\mathbf{\frac{2}{3}(mp)^3}$ flops. The extraction of the minimal realization needs $\mathbf{2(mp)^3+2(mp)^2}$ flops.

### 6.2.7 Observable realization reduction algorithm:

By duality, this algorithm starts by an observable realization of dimension *mq*x*mq*, then the controllability matrix is constructed and searched to obtain the similarity matrix so that the original realization is reduced to a minimal one (see chapter5 section5.3.2).

**cost:** The observable realization $\{A,B,C\}$ is of dimension $qm\text{x}qm$, $qm\text{x}p$ and $q\text{x}qm$ respectively. The controllability matrix is of dimension $qm\text{x}p(m\text{-}1)$ and uses $((\mathbf{qm})^2\mathbf{p})(\mathbf{m}-2)$ flops. Then the recursive algorithm is used to search this matrix and it is transformed to a similarity matrix of dimension $qm\text{x}qm$. Its inverse is computed using $\frac{2}{3}(\mathbf{qm})^3$ flops. The extraction of the irreducible realization uses $2(\mathbf{qm})^3 + 2(\mathbf{qm})^2$ flops.

## 6.3 NUMERICAL RESULTS

To provide a reliable test of the proposed algorithms, a quite large number of transfer function matrices have been used. Some of them have been chosen especially to show a particular property of the algorithms; others have been used randomly.

### 6.3.1 Example 1:

The first transfer function matrix has been used by Chen [6] to illustrate his method. We have given details on how the recursive algorithm works.

$$\text{Consider: } \hat{G}(s) = \begin{bmatrix} \dfrac{-2s^2-3s-2}{(s+1)^2} & \dfrac{1}{s^2} \\ \dfrac{4s+5}{s+1} & \dfrac{-3s-5}{s+2} \end{bmatrix}$$

Computing the Markov parameters we get:

$$\frac{-2s^2-3s-2}{(s+1)^2} = -2 + s^{-1} - 2s^{-2} + 3s^{-3} - 4s^{-4} + 5s^{-5} - 6s^{-6} + \ldots$$

$$\frac{1}{s^2} = 0 + 0s^{-1} + s^{-2} + 0s^{-3} + \ldots$$

$$\frac{4s+5}{s+1} = 4 + s^{-1} - s^{-2} + s^{-3} - s^{-4} + \ldots$$

$$\frac{-3s-5}{s+2} = -3 + s^{-1} - 2s^{-2} + 4s^{-3} - 8s^{-4} + 16s^{-5} + \dots$$

we have $\begin{cases} \alpha_1 = 4 \\ \alpha_2 = 2 \end{cases}$ and $\begin{cases} \beta_1 = 2 \\ \beta_2 = 3 \end{cases}$. The Hankel matrix is given by:

$$T = \begin{bmatrix} H_{11}(\alpha_1+1,\beta_1) & H_{12}(\alpha_1+1,\beta_2) \\ H_{21}(\alpha_2+1,\beta_1) & H_{22}(\alpha_2+1,\beta_2) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -2 & \vdots & 0 & 1 & 0 \\ -2 & 3 & \vdots & 1 & 0 & 0 \\ 3 & -4 & \vdots & 0 & 0 & 0 \\ -4 & 5 & \vdots & 0 & 0 & 0 \\ 5 & -6 & \vdots & 0 & 0 & 0 \\ \dots & \dots & \vdots & \dots & \dots & \dots \\ 1 & -1 & \vdots & 1 & -2 & 4 \\ -1 & 1 & \vdots & -2 & 4 & -8 \\ 1 & -1 & \vdots & 4 & -8 & 16 \end{bmatrix}$$

we search the rows of T using the recursive algorithm.

-P(0) is initialised: $P(0) = I_5$

-For k=1,...,8 and taking the rows of T as vectors $V_i$ we compute:

$$* \; V_1^T P(0) V_1 = 6 \Rightarrow V_1 \neq 0 \text{ and } P(1) = \begin{bmatrix} 0.8333 & 0.3333 & 0 & -0.1667 & 0 \\ 0.3333 & 0.3333 & 0 & 0.3333 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.1667 & 0.3333 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$*V_2^T P(1) V_2 = 3.3333$ then $V_2$ is linearly independent of $V_1$

$$\text{and } P(2) = \begin{bmatrix} 0.7 & 0.4 & 0.2 & 0.1 & 0 \\ 0.4 & 0.3 & -0.1 & 0.2 & 0 \\ 0.2 & -0.1 & 0.7 & 0.4 & 0 \\ 0.1 & 0.2 & -0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This process is continued until the fifth row

$^{*}V_5^T P(4) V_5 = 5.5511e^{-016} \approx 0$ then $V_5$ is a primary linearly dependent vector of the independent ones $\{V_1, V_2, V_3, V_4\}$,

$$\text{and } P(5) = P(4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and so on...}$$

The algorithm yields that the 5th, 7th and 8th rows are linearly dependent of the previous ones. Using the improved Gauss-Jordan method to solve for the coefficients we get for the primary dependent rows:

$$\begin{cases} row5 = 0 \times row1 + 0 \times row2 - row3 - 2 \times row4 \\ row7 = 0 \times row1 + 0 \times row2 - row3 - row4 + 0 \times row5 - 2 \times row6 \end{cases}$$

The minimal realization will be:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & 0 \\ -4 & 0 \\ 1 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 4 & -3 \end{bmatrix} u \end{cases}$$

### 6.3.2 Example 2:

This example, given by Mayne in reference[19], is to illustrate the second method proposed by Chen and Mital [6] and modified by the introduction of the recursive algorithm. So we consider the following simple system:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

computing the Markov parameters we get: $\frac{1}{s+1} = s^{-1} - s^{-2} + s^{-3} + \ldots$

So G(s) is rewritten as: $G(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} s^{-1} + \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} s^{-2} + \ldots$

we have also: $\boldsymbol{\alpha} = 1,\ \boldsymbol{\beta} = 1$

The first Hankel matrix is: $\overline{T} = \begin{bmatrix} H(1) \\ \hline H(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \hline -1 & -1 \\ -1 & -1 \end{bmatrix}$

This matrix is searched using the recursive algorithm and it is found that the last three rows are linearly dependent, but the primary ones are only the second and the third rows.so the coefficients of combination are:

$$k_1 = [-1 \quad (1) \ \vdots \ 0 \quad 0]$$
$$k_2 = [\ 1 \quad 0 \ \vdots \ (1) \quad 0]$$

Rearranging the Hankel matrix we get: $\hat{T} = \begin{bmatrix} H_{11}(2,1) & H_{12}(2,1) \\ H_{21}(2,1) & H_{22}(2,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ \hline 1 & 1 \\ -1 & -1 \end{bmatrix}$

and $\begin{matrix} \hat{k}_1 = [-1 \quad 0 \ \vdots \ (1) \quad 0] \\ \hat{k}_2 = [\ 1 \quad (1) \ \vdots \ 0 \quad 0] \end{matrix}$

Thus obtaining for the minimal realization: $\begin{cases} \dot{\mathbf{x}} = -\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} \end{cases}$

### 6.3.3 Example 3:

This example is from reference[10] and the controllable realization reduction will be used.

We'll look for the minimal realization of G(s):

$$G(s) = \frac{1}{s(s+1)(s+2)} \begin{bmatrix} 2(2s^2+4s+1) & s(s+2) \\ 2s(3s+5) & (s+2)(3s+1) \end{bmatrix}$$

**a_Controllable realization:**

we rewrite G(s) as: $G(s) = \dfrac{1}{s^3+3s^2+2s+0} \left[ \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix} s^2 + \begin{pmatrix} 8 & 2 \\ 10 & 7 \end{pmatrix} s + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right]$

the controllable realization is: $\begin{cases} \dot{x} = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & -2I_2 & -3I_2 \end{bmatrix} x + \begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix} u \\ y = \begin{bmatrix} 2 & 0 & 8 & 2 & 4 & 1 \\ 0 & 2 & 10 & 7 & 6 & 3 \end{bmatrix} x \end{cases}$

**b_ Reduction process:**

1_ Observability matrix:

$$O(C,A) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 8 & 2 & 4 & 1 \\ 0 & 2 & 10 & 7 & 6 & 3 \\ 0 & 0 & -6 & -2 & -4 & -1 \\ 0 & 0 & -12 & -4 & -8 & -2 \\ 0 & 0 & 8 & 2 & 6 & 1 \\ 0 & 0 & 16 & 4 & 12 & 2 \end{bmatrix}$$

2_ O(C,A) is searched using the recursive algorithm:

_P(0) is initialised: $I_6$

-For k=1,...,6 and taking the rows of O as vectors $V_i$ , we compute:

* $V_1^T P(0) V_1 = 89 \Rightarrow V_1 \neq 0$ and

$$P(1) = \begin{bmatrix} 0.9551 & 0 & -0.1798 & -0.0449 & -0.0899 & -0.0225 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.1798 & 0 & 0.2809 & -0.1798 & -0.3596 & -0.0899 \\ -0.0449 & 0 & -0.1798 & 0.9551 & -0.0899 & -0.0225 \\ -0.0899 & 0 & -0.3596 & -0.0899 & 0.8202 & -0.0449 \\ -0.0225 & 0 & -0.0899 & -0.0225 & -0.0449 & 0.9888 \end{bmatrix}$$

*$V_2^T P(1) V_2 = 33.4944$ then $V_2$ is linearly independent of $V_1$

$$\text{and } P(2) = \begin{bmatrix} 0.7343 & 0.1624 & -0.2509 & 0.3026 & -0.0443 & 0.1107 \\ 0.1624 & 0.8806 & 0.0523 & -0.2556 & -0.0335 & -0.0980 \\ -0.2509 & 0.0523 & 0.0258 & -0.0678 & -0.3449 & -0.0470 \\ 0.3026 & -0.2556 & -0.0678 & 0.4079 & -0.1617 & -0.2321 \\ -0.0443 & -0.0335 & -0.3449 & -0.1617 & 0.8108 & -0.0725 \\ 0.1107 & -0.0980 & -0.0470 & -0.2321 & -0.0725 & 0.9084 \end{bmatrix}$$

The process is continued until the fourth row

*$V_4^T P(3) V_4 = -3.3307e^{-016} \approx 0$ then $V_4$ is a primary linearly dependent vector of the independent ones $\{V_1, V_2, V_3\}$, and

$$P(4) = P(3) = \begin{bmatrix} 0.2579 & -0.0492 & -0.2579 & 0.0711 & 0.3393 & 0.0479 \\ -0.0492 & 0.7866 & 0.0492 & -0.3584 & 0.1368 & -0.1259 \\ -0.2579 & 0.0492 & 0.2579 & -0.0711 & -0.3393 & -0.0479 \\ 0.0711 & -0.3584 & -0.0711 & 0.2955 & 0.0246 & -0.2627 \\ 0.3393 & 0.1368 & -0.3393 & 0.0246 & 0.5021 & -0.0219 \\ 0.0479 & -0.1259 & -0.0479 & -0.2627 & -0.0219 & 0.9001 \end{bmatrix}$$

Following the process we get that the fourth and sixth rows are linearly dependent.

3_ Computation of new linear independent rows to get a non-singular T:

new 4th row=(0 0 144 16 64 4)

new 6th row=(0 0 256 16 144 4)

$$\text{so } T = \begin{bmatrix} 2 & 0 & 8 & 2 & 4 & 1 \\ 0 & 2 & 10 & 7 & 6 & 3 \\ 0 & 0 & -6 & -2 & -4 & -1 \\ 0 & 0 & 8 & 2 & 6 & 1 \\ 0 & 0 & 144 & 16 & 64 & 4 \\ 0 & 0 & 256 & 16 & 144 & 4 \end{bmatrix}$$

4_ Using the similarity transformation we get a new realization:

$$\overline{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 & 0 & 0 \\ 0 & 0 & -66 & -54 & -3.75 & 1.75 \\ 0 & 0 & -90 & -78 & -2.75 & 0.75 \end{bmatrix}, \overline{B} = \begin{bmatrix} 4 & 1 \\ 6 & 3 \\ -4 & -1 \\ 6 & 1 \\ 64 & 4 \\ 144 & 4 \end{bmatrix}, \overline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5_ Extracting the minimal realization:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 4 & 1 \\ 6 & 3 \\ -4 & -1 \\ 6 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \end{cases}$$

### 6.3.4 Example 4:

This example is from reference[10] and illustrates the same algorithm as the previous. So we will look for the minimal realization of G(s) given by:

$$G(s)=\frac{1}{s^3+2s^2+2s+1}\begin{bmatrix} s+2 & s^2+4s+3 \\ 2s+3 & 2s^2+7s+5 \end{bmatrix}$$

**a_Controllable realization:**

G(s) is rewritten as: $G(s)=\dfrac{1}{s^3+2s^2+2s+1}\left[\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}s^2+\begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}s+\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right]$

the controllable realization is: $\begin{cases} \dot{x}=\begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -I_2 & -2I_2 & -2I_2 \end{bmatrix}x+\begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix}u \\ y=\begin{bmatrix} 2 & 3 & 1 & 4 & 0 & 1 \\ 3 & 5 & 2 & 7 & 0 & 2 \end{bmatrix}x \end{cases}$

**b_ Reduction process:**

observability matrix: $O(C,A)=\begin{bmatrix} 2 & 3 & 1 & 4 & 0 & 1 \\ 3 & 5 & 2 & 7 & 0 & 2 \\ 0 & -1 & 2 & 1 & 1 & 2 \\ 0 & -2 & 3 & 1 & 2 & 3 \\ -1 & -2 & -2 & -5 & 0 & -3 \\ -2 & -3 & -4 & -8 & -1 & -5 \end{bmatrix}$

This matrix is searched using the recursive algorithm and it is found that the last row is linearly dependent of the previous rows, which means that the minimal dimension of this system is 5. So computing the last independent row we get for the similarity transformation T :

$$T=\begin{bmatrix} 2 & 3 & 1 & 4 & 0 & 1 \\ 3 & 5 & 2 & 7 & 0 & 2 \\ 0 & -1 & 2 & 1 & 1 & 2 \\ 0 & -2 & 3 & 1 & 2 & 3 \\ -1 & -2 & -2 & -5 & 0 & -3 \\ 4 & 9 & 16 & 64 & 1 & 25 \end{bmatrix}$$

A new realization is obtained from which we extract the following irreducible realization:

$$\begin{cases} \dot{\mathbf{x}}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & -2 & 0 & -2 \end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 2 \\ 2 & 3 \\ 0 & -3 \end{bmatrix}\mathbf{u} \\ \mathbf{y}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x} \end{cases}$$

### 6.3.5 Example 5:

This example is from reference [10] and the observable realization reduction is used. The following transfer function matrix is considered:

$$G(s)=\frac{1}{(s+1)(s+2)(s+3)}\begin{bmatrix} 6s^2+21s+17 & 4s^2+14s+10 \\ 14s^2+49s+41 & 7s^2+23s+16 \end{bmatrix}$$

1 _ G(s) will be written as: $\begin{cases} G(s)=\begin{bmatrix} 6 & 4 \\ 14 & 7 \end{bmatrix}s^{-1}+\begin{bmatrix} -15 & -10 \\ -35 & -19 \end{bmatrix}s^{-2}+\begin{bmatrix} 41 & 26 \\ 97 & 53 \end{bmatrix}s^{-3}+\ldots \\ \boldsymbol{\Psi}(s)=s^3+6s^2+11s+6 \end{cases}$

Then the observable realization is:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -6I_2 & -11I_2 & -6I_2 \end{bmatrix} x + \begin{bmatrix} 6 & 4 \\ 14 & 7 \\ -15 & -10 \\ -35 & -19 \\ 41 & 26 \\ 97 & 53 \end{bmatrix} u \\ y = \begin{bmatrix} I_2 & 0_2 & 0_2 \end{bmatrix} x + H(0)u \end{cases}$$

2_ The controllability matrix is: $\mathbf{Cn(A,B)} = \begin{bmatrix} \mathbf{6} & \mathbf{4} & \mathbf{-15} & \mathbf{-10} & \mathbf{41} & \mathbf{26} \\ \mathbf{14} & \mathbf{7} & \mathbf{-35} & \mathbf{-19} & \mathbf{97} & \mathbf{53} \\ \mathbf{-15} & \mathbf{-10} & \mathbf{41} & \mathbf{26} & \mathbf{-117} & \mathbf{-70} \\ \mathbf{-35} & \mathbf{-19} & \mathbf{97} & \mathbf{53} & \mathbf{-281} & \mathbf{-151} \\ \mathbf{41} & \mathbf{26} & \mathbf{-117} & \mathbf{-70} & \mathbf{341} & \mathbf{194} \\ \mathbf{97} & \mathbf{53} & \mathbf{-281} & \mathbf{-151} & \mathbf{829} & \mathbf{437} \end{bmatrix}$

3_ Using the recursive algorithm we get:

_P(0) is initialised: $I_6$

-For k=1,....,6 and taking the columns of Cn as vectors $V_i$ we compute:

* $V_1^T P(0) V_1 = 12772 \Rightarrow V_1 \neq 0$ and,

$$P(1) = \begin{bmatrix} 0.9972 & -0.0066 & 0.0070 & 0.0164 & -0.0193 & -0.0456 \\ -0.0066 & 0.9847 & 0.0164 & 0.0384 & -0.0449 & -0.1063 \\ 0.0070 & 0.0164 & 0.9824 & -0.0411 & 0.0482 & 0.1139 \\ 0.0164 & 0.0384 & -0.0411 & 0.9041 & 0.1124 & 0.2658 \\ -0.0193 & -0.0449 & 0.0482 & 0.1124 & 0.8684 & -0.3114 \\ -0.0456 & -0.1063 & 0.1139 & 0.2658 & -0.3114 & 0.2633 \end{bmatrix}$$

* $V_2^T P(1) V_2 = 15.0138$ then $V_2$ is linearly independent of $V_1$ and,

$$P(2)=\begin{bmatrix} 0.9696 & 0.0291 & 0.0761 & -0.0083 & -0.1508 & 0.0083 \\ 0.0291 & 0.9387 & -0.0726 & 0.0703 & 0.1248 & -0.1759 \\ 0.0761 & -0.0726 & 0.8098 & 0.0208 & 0.3778 & -0.0208 \\ -0.0083 & 0.0703 & 0.0208 & 0.8819 & -0.0055 & 0.3141 \\ -0.1508 & 0.1248 & 0.3770 & -0.0055 & 0.2420 & -0.0547 \\ 0.0083 & -0.1759 & -0.0208 & 0.3141 & -0.0547 & 0.1581 \end{bmatrix}$$

The process is continued until the fourth column

$*V_4^T P(3) V_4 = 3.8559e^{-14} \approx 0$ then $V_4$ is a primary linearly dependent vector of the independent ones $\{V_1, V_2, V_3\}$ and ,

$$P(4)=P(3)=\begin{bmatrix} 0.8918 & -0.1747 & 0.1379 & 0.1267 & -0.1456 & 0.0986 \\ -0.1747 & 0.4052 & 0.0893 & 0.4239 & 0.1384 & 0.0606 \\ 0.1379 & 0.0893 & 0.7606 & -0.0866 & 0.3728 & -0.0926 \\ 0.1267 & 0.4239 & -0.0866 & 0.6475 & -0.0146 & 0.1574 \\ -0.1456 & 0.1384 & 0.3728 & -0.0146 & 0.2416 & -0.0607 \\ 0.0986 & 0.0606 & -0.0926 & 0.1574 & -0.0607 & 0.0533 \end{bmatrix}$$

Following the process we get that the fourth, fifth and sixth columns are linearly dependent.

4_ Computing the 3 new linear independent columns to construct a non-singular similarity transformation matrix we get:

new 4th column=(100,361,676,2809,4900,22801)'

new 5th column=(1681,9409,13689,78961,116281,687241)'

new 6th column=(676,2809,4900,22801,37636,190969)'

5_ Performing the similarity transformation we get:

$$\bar{A}=\begin{bmatrix} 0 & 0.2857 & -3.1429 & -22700 & -1030000 & -237500 \\ 0 & -1.8571 & -0.5714 & 3600 & 447100 & 34400 \\ 1 & 0.2857 & -4.1429 & -7600 & -349300 & -76500 \\ 0 & 0 & 0 & 3300 & 134300 & 33500 \\ 0 & 0 & 0 & 100 & 2900 & 700 \\ 0 & 0 & 0 & -700 & -26600 & -6200 \end{bmatrix}$$

$$\bar{B}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{C}=\begin{bmatrix} 6 & 4 & -15 & 100 & 1681 & 676 \\ 14 & 7 & -35 & 361 & 9409 & 2809 \end{bmatrix}$$

6_ Extracting the minimal realization:

$$\begin{cases} \dot{x}=\begin{bmatrix} 0 & 0.2857 & -3.1429 \\ 0 & -1.8571 & -0.5714 \\ 1 & 0.2857 & -4.1429 \end{bmatrix}x+\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}u \\ y=\begin{bmatrix} 6 & 4 & -15 \\ 14 & 7 & -35 \end{bmatrix}x \end{cases}$$

### 6.3.6 Example 6:

This example is from reference [10] and it illustrates the same algorithm as the previous. We consider the following transfer function matrix:

$$G(s)=\frac{1}{s^3+3s^2+3s+1}\begin{bmatrix} 2s^2+4s+7 & s^2+2s+4 \\ 4s+7 & s+3 \end{bmatrix}$$

It is rewritten as: $G(s)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}s^{-1}+\begin{bmatrix} -2 & -1 \\ 4 & 1 \end{bmatrix}s^{-2}+\begin{bmatrix} 7 & 4 \\ -5 & 0 \end{bmatrix}s^{-3}+\ldots$

Then the observable realization is:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -I_2 & -3I_2 & -3I_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ -2 & -1 \\ 4 & 1 \\ 7 & 4 \\ -5 & 0 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} I_2 & 0_2 & 0_2 \end{bmatrix} \mathbf{x} \end{cases}$$

The controllability matrix is: $\mathbf{Cn(A,B)} = \begin{bmatrix} 2 & 1 & -2 & -1 & 7 & 4 \\ 0 & 0 & 4 & 1 & -5 & 0 \\ -2 & -1 & 7 & 4 & -17 & -10 \\ 4 & 1 & -5 & 0 & 3 & -3 \\ 7 & 4 & -17 & -10 & 32 & 19 \\ -5 & 0 & 3 & -3 & 2 & 8 \end{bmatrix}$

Using the recursive algorithm to search the columns of Cn(A,B), we found that the last column is linearly dependent of the previous ones. This means that the minimal dimension of this system is 5. Computing the inedependent column we get: col6=( 16 0 100 9 361 64)'.

Performing the operations to get a new realization, and then extracting a minimal realization we get:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0.6969 & -1 \\ 0 & 0 & 0 & -1.1607 & 0 \\ 1 & 0 & 0 & 1.2847 & -3 \\ 0 & 1 & 0 & -2.1446 & 0 \\ 0 & 0 & 1 & 0.5988 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} 2 & 1 & -2 & -1 & 7 \\ 0 & 0 & 4 & 1 & -5 \end{bmatrix} \mathbf{x} \end{cases}$$

# CHAPTER 7

## CONCLUSION

The importance of the minimal realization has been emphasized by many authors. The minimal realization is the state-vector model of lowest dimension having a specified matrix transfer function; reduction of a non minimal realization simplifies subsequent control and filtering algorithms.
A procedure for obtaining minimal realizations is also required by some algorithms for spectral factorization, which is essentially the problem of obtaining a minimal dimension state-vector model which, if excited by white noise, yields a specified spectral density matrix.

The contribution of this thesis is the presentation of three methods of minimal realization of systems described by rational transfer function matrices using a recursive algorithm based on orthogonally projected vectors.

The recursive algorithm is efficient and stable since it uses the principle of orthogonality which keeps the norm of the vectors constant, thus avoiding to change the condition of the matrices. In addition it is easily computerized, saves space since it uses a single matrix and saves time since it is recursive and avoids matrix inversion.

This algorithm, which consists of searching in a set of vectors linear independent ones, is important in the sense that it can be used for other purposes in many domains.

The first method presented in this thesis is nearly the same as the one presented by Chen and Mital except for the use of the recursive algorithm.
The main advantage claimed for this method is that the realization is obtained in a canonical form, and this fact is retained by the proposed method.
Comparing the two methods, we can conclude that the proposed one saves space since it uses one matrix instead of two of the same dimension used by Chen. It saves time since the order of computations in Chen's method is $\mathbf{O(n^4)}$ and in the proposed method it is $\mathbf{O(n^3)}$.
The only disadvantage of the proposed method is in the computation of the coefficients of combination which is of order $\mathbf{O(n^3)}$ compared to Chen's method which is of order $\mathbf{O(\frac{1}{2}n^2)}$. But still, the use of the improved Gauss-Jordan method assures us efficiency and stability.

Comparing these two methods with the singular value decomposition method proposed also by Chen and Mital, we can say that the latter is the worse since it uses more space and time than the others.

The second method proposed in this thesis is simple since the main operation, which is the searching for linear independent rows is done using the recursive algorithm. In addition, the controllable realization is obtained directly which minimizes the number of operations to be executed by the computer. Besides this, C is obtained in a canonical form thus reducing the analog circuit considerably.

It is complete since it gives the original matrix to be reduced and a simple algorithm to construct the similarity transformation matrix.
Its disadvantage, compared to Chen's method for example, is the use of large sparse matrices such as the controllable realization and the observability matrix. But these matrices are meant to be reduced, and their being sparse can be taken advantage of by using programming techniques related to sparse matrices.

Dual to the second method, the third method is simple and the main operation, the column-searching, is done using the recursive algorithm. The observable realization is easily obtained, and computation of new linear independent vectors is done using the linear dependent ones. As for the previous method, B is obtained in a canonical form which requires a relatively small number of amplifiers and potentiometers for simulation on analog computers. Compared to Kalman method, it is complete and its efficiency comes from the use of the recursive algorithm.
Its disadvantage is, as for the precedent, the use of large sparse matrices as the observable realization and the controllability matrix.

Comparing the three methods, the modified Chen's method (first) is more valuable in terms of computer programming because the minimal realization is obtained in canonical form and the recursive algorithm and the algorithms to compute Markov parameters and coefficients of combinations are easily computerized. Using computer optimization techniques we can achieve an efficient program to give directly the minimal realization of any system.

As suggestions for further research, a more efficient algorithm to compute the coefficients of combination of linearly dependent rows of the Hankel matrix in the first method can be found to reduce the cost of the proposed algorithm even more than it is now.

It is also suggested to investigate better ways of constructing new linearly independent vectors to complete the partial set in order to construct the similarity transformation matrix in the second and third methods.

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