Numerical Solution for Nonlinear Time-Fractional Partial Differential Equation With Variable Coefficient Using Reproducing Kernel Hibert Space Method

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***Abstract*—The iterative reproducing kernel Hilbert space method is used to approximate the solution of the nonlinear time-fractional partial differential equation with variable coefficient. The experiment results show the rapidity of convergence of the approximate solution and demonstrate that this method is effective and accurate method.**

***Index Terms*—Fractional Calculus, Caputo Derivative, Reproducing Kernel Hilbert Space Method, Partial Differential Equation, Gram-Schmidt process, Numerical solution, Convergence.**

I. Introduction

The fractional calculus is well known as derivations and integrations of arbitrary order theory in modern mathematics. The very important role of the fractional calculus is the generalization of n-fold integration (or integer order differentiation) notion to be non-integer order.

Recently, an important phenomena can be achievely modeled by fractional partial differential equations in engineering and sciences such as diffusion wave, control theory, dynamical systems, fluid mechanics and many other subjects.

In this research, nonlinear time-fractional partial differential equation based on the fractional derivative in the sense of Caputo given by (1) is solved by the reproducing kernel Hilbert space method.

$$\begin{cases} F(\zeta,\eta,\vartheta,\vartheta_\zeta,\mathfrak{D}_\eta^\alpha\vartheta,\dots)=\hbar(\zeta,\eta),\ 0<\zeta,\eta<1,\ 0<\alpha\leq 1,\\ \vartheta(\zeta,0)=0,\quad 0\leq\zeta\leq 1,\\ \vartheta(0,\eta)=\vartheta(1,\eta)=0\quad 0\leq\eta\leq 1. \end{cases} \tag{1}$$

where $\hbar(\zeta,\eta)$ is continuous function and $\mathfrak{D}_\eta^\alpha$ is the fractional derivative describing the Caputo sense with respect to time η.

Several methods have been used to study partial differential equations described by fractional-order, including the finite difference method, the finite element method, the collocation method and the Adomian's decomposition method.

The concept of reproducing kernel Hilbert space (RKHS) method (Cui and Lin, 2009) will be effectively used to give the solution of fractional partial differential equation in convergent series form.

This paper is structured as follows. In Section 2, we provide some important definitions and mathematical premilinaries of the reproducing kernel theory and the fractional calculus theory. In Section 3, we present the application of the reproduction kernel Hilbert space method to construct the solution of our model problem. In Section 4, we apply the proposed method to two numerical examples for demonstrate the accuracy and applicability of this method.

II. Some Basic Preliminary

We introduce in this section some basic definitions and concepts from reproducing kernel theory which are important for other sections of this study.

Definition 2.1: The fractional derivative in Caputo sense of order $\varsigma>0$ is

$$(\mathfrak{D}_\eta^\varsigma\vartheta)(\zeta,\eta)=\frac{1}{\Gamma(m-\varsigma)}\int_0^\eta(\eta-\mu)^{m-\varsigma-1}\partial_\mu^m\vartheta(\zeta,\mu)d\mu, \tag{2}$$

for $\eta>0,\ m-1<\varsigma\leq m$ and $m\in\mathbb{N}$.

Definition 2.2: Let a Hilbert space $\mathfrak{H}$. We call a function $K_\rho(\cdot)$, where $K:\chi\times\chi\to\mathbb{C}$ a reproducing kernel of $\mathfrak{H}$, if it satisfies the following two conditions

1) $K_\rho(\cdot)\in\mathfrak{H},\ \forall\rho\in\mathcal{X}$.

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2) $\langle \vartheta(\cdot), K_\rho(\cdot)\rangle = \vartheta(\rho)$ for all $\vartheta \in \mathfrak{H}$ and $\rho \in \chi$.

A. The Reproducing Kernel Hilbert Space $W_2^3[0,1]$

$W_2^3[0,1]$ is a space of functions which is defined as follows $W_2^3[0,1] = \{\Theta | \Theta^{(\ell)}(\zeta)$ are absolutely continuous functions on $[0,1], \Theta^{(3)}(\zeta) \in L^2[0,1], \ell = 0,1,2,$ and $\Theta(0) = \Theta'(0) = 0\}$. With its corresponding inner product

$$\langle \Theta, \Upsilon\rangle_{W_2^3} = \sum_{\ell=0}^{2} \Theta^{(\ell)}(0)\Upsilon^{(\ell)}(0) + \int_0^1 \Theta^{(3)}(\zeta)\Upsilon^{(3)}(\zeta)d\zeta$$

and its corresponding norm $\|\Theta\|_{W_2^3} = \langle \Theta, \Theta\rangle_{W_2^3}^{\frac{1}{2}}$ in which $\Theta, \Upsilon \in W_2^3[0,1]$.

The reproducing Kernel function $K_\zeta(\kappa)$ associated to $W_2^3[0,1]$ given by

$$K_\zeta(\kappa) = \begin{cases} k(\zeta,\kappa) & ,\kappa \le \zeta, \\ k(\kappa,\zeta) & ,\kappa > \zeta. \end{cases} \tag{3}$$

where $k(\zeta,\kappa) = (\kappa^2/120)(-5\zeta\kappa^2 + \kappa^3 + 10\zeta^2(3+\kappa))$.

B. The Reproducing Kernel Hilbert Space $W_2^2[0,1]$

$W_2^2[0,1]$ is a space of functions which is defined as follows $W_2^2[0,1] = \{\Theta | \Theta^{(\ell)}(\eta)$ are absolutely continuous functions on $[0,1], \Theta''(\eta) \in L^2[0,1], \ell = 0,1,$ and $\Theta(0) = 0\}$.

With its corresponding inner product

$$\langle \Theta, \Upsilon\rangle_{W_2^2} = \sum_{\ell=0}^{1} \Theta^{(\ell)}(0)\Upsilon^{(\ell)}(0) + \int_0^1 \Theta''(\eta)\Upsilon''(\eta)d\eta$$

and its corresponding norm $\|\Theta\|_{W_2^2} = \langle \Theta, \Theta\rangle_{W_2^2}^{\frac{1}{2}}$ in which $\Theta, \Upsilon \in W_2^2[0,1]$.

The reproducing Kernel function $S_\eta(\nu)$ associated to the space $W_2^2[0,1]$, given by

$$S_\eta(\nu) = \begin{cases} s(\eta,\nu) & ,\nu \le \eta, \\ k(\nu,\eta) & ,\nu > \eta. \end{cases} \tag{4}$$

where $k(\eta,\nu) = (\nu/6)(-\nu^2 + 3\eta(2+\nu))$.

C. The Reproducing Kernel Hilbert Space $W_2^1[0,1]$

$W_2^1[0,1]$ is a space of functions which is defined as follows $W_2^1[0,1] = \{\Theta | \Theta(\zeta)$ is absolutely continuous functions on $[0,1], \Theta'(\zeta) \in L^2[0,1]\}$.

With its corresponding inner product

$$\langle \Theta, \Upsilon\rangle_{W_2^1} = \int_0^1 (\Theta(\zeta)\Upsilon(\zeta) + \Theta'(\zeta)\Upsilon'(\zeta))d\zeta$$

and its corresponding norm $\|\Theta\|_{W_2^1} = \sqrt{\langle \Theta, \Theta\rangle_{W_2^1}}$ in which $\Theta, \Upsilon \in W_2^1[0,1]$.

The reproducing kernel function $R_\zeta(\kappa)$ associated to the complete reproducing kernel space $W_2^1[0,1]$ is given by

$$R_\zeta(\kappa) = \begin{cases} 1+\kappa & ,\kappa \le \zeta, \\ 1+\zeta & ,\kappa > \zeta. \end{cases} \tag{5}$$

D. The Binary Reproducing Kernel Hilbert Space $W_2^{(1,1)}(F)$

Let be $F = [0,1] \times [0,1]$.

$W_2^{(1,1)}(F)$ is a space of functions which is defined as follows $W_2^{(1,1)}(F) = \{\vartheta(\zeta,\eta) | \vartheta$ are absolutely continuous functions on $F, \frac{\partial^2\vartheta}{\partial\zeta\partial\eta} \in L^2(F)\}$.

With its corresponding inner product

$$\langle \vartheta(\zeta,\eta), v(\zeta,\eta)\rangle_{W_2^{(1,1)}(F)} = \int_0^1 \left[\frac{\partial}{\partial\eta}\vartheta(0,\eta)\frac{\partial}{\partial t}v(0,\eta)\right]d\eta$$
$$+ \langle \vartheta(\zeta,0), v(\zeta,0)\rangle_{W_2^1} + \int_0^1\int_0^1 \frac{\partial}{\partial\zeta}\frac{\partial}{\partial\eta}\vartheta(\zeta,\eta)\frac{\partial}{\partial\zeta}\frac{\partial}{\partial}v(\zeta,\eta)d\zeta d\eta$$

and its corresponding norm $\|\vartheta\|_{W_2^{(1,1)}(F)} = \sqrt{\langle \vartheta, \vartheta\rangle_{W_2^{(1,1)}(F)}}$ in which $\vartheta, v \in W_2^{(1,1)}(F)$.

The reproducing Kernel function $\tilde{Q}_{(\zeta,\eta)}(\kappa,\mu)$ associated to the space $W_2^{(1,1)}(F)$, given by

$$\tilde{Q}_{(\zeta,\eta)}(\kappa,\mu) = R_\zeta(\kappa)R_\eta(\mu) \tag{6}$$

E. The Binary Reproducing Kernel Hilbert Space $W_2^{(3,2)}(F)$

$W_2^{(3,2)}(F)$ is a space of functions which is defined as follows $W_2^{(3,2)}(F) = \{\vartheta(\zeta,\eta) | \frac{\partial^3\vartheta}{\partial\zeta^2\partial\eta}$ are absolutely continuous functions on $F, \frac{\partial^5\vartheta}{\partial\zeta^3\partial\eta^2} \in L^2(F)$ and, $\vartheta(\zeta,0) = \vartheta(0,\eta) = \vartheta(1,\eta) = 0\}$.

With its corresponding inner product

$$\langle \vartheta(\zeta,\eta), v(\zeta,\eta)\rangle_{W_2^{(3,2)}(F)} =$$
$$\sum_{\iota=0}^{1}\int_0^1 \left[\frac{\partial^2}{\partial\eta^2}\frac{\partial^\iota}{\partial\zeta^\iota}\vartheta(0,\eta)\frac{\partial^2}{\partial\eta^2}\frac{\partial^\iota}{\partial x^\iota}v(0,\eta)\right]d\eta$$
$$+ \int_0^1 \left[\frac{\partial^2}{\partial\eta^2}\vartheta(1,\eta)\frac{\partial^2}{\partial\eta^2}v(1,\eta)\right]d\eta$$
$$+ \sum_{\ell=0}^{1}\left\langle \frac{\partial^\ell}{\partial\eta^\ell}\vartheta(\zeta,0), \frac{\partial^\ell}{\partial\eta^\ell}v(\zeta,0)\right\rangle_{W_2^m}$$
$$+ \int_0^1\int_0^1 \frac{\partial^3}{\partial\zeta^3}\frac{\partial^2}{\partial\eta^2}\vartheta(\zeta,\eta)\frac{\partial^3}{\partial\zeta^3}\frac{\partial^2}{\partial\eta^2}v(\zeta,\eta)d\zeta d\eta$$

and its corresponding norm $\|\vartheta\|_{W_2^{(3,2)}(F)} = \sqrt{\langle \vartheta, \vartheta\rangle_{W_2^{(3,2)}(F)}}$ in which $\vartheta, v \in W_2^{(3,2)}(F)$.

The reproducing Kernel function $Q_{(\zeta,\eta)}(\kappa,\mu)$ associated to the space $W_2^{(3,2)}(F)$, given by

$$Q_{(\zeta,\eta)}(\kappa,\mu) = K_\zeta(\kappa)S_\eta(\mu) \tag{7}$$

III. Analysis of the Reproducing Kernel Hilbert Space Method

Firstly, we we can rewrite the problem (1) as follows

$$\begin{cases} \Im\vartheta(\zeta,\eta) = G(\zeta,\eta,\vartheta,\vartheta_\zeta,\dots),\ 0<\zeta,\eta<1,\ 0<\alpha\le 1, \\ \vartheta(\zeta,0) = 0, \quad 0\le\zeta\le 1, \\ \vartheta(0,\eta) = \vartheta(1,\eta) = 0 \quad 0\le\eta\le 1. \end{cases} \tag{8}$$

where $\Im : W_2^{(3,2)}(F) \to W_2^{(1,1)}(F)$ is a linear operator.

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Secondly, to construct the base of $W_2^{(3,2)}(F)$ which is called the orthogonal function system, let $\wp_\iota(\zeta,\eta) = R_{(\zeta_\iota,\eta_\iota)}(\zeta,\eta)$ and $\psi_\iota(\zeta,\eta) = \Im^*\wp_\iota(\zeta,\eta)$, where $\{(\zeta_\iota,\eta_\iota)\}_{\iota=1}^{\infty}$ form a countable set which is dense on interval F, $\Im^*$ denotes the formal adjoint operator of $\Im$ and the reproducing kernel function associated to $W_2^{(1,1)}(F)$ is denoted by $R_{(\zeta_\iota,\eta_\iota)}(\zeta,\eta)$

Next, using the process of Gram-Schmidt, we obtain an orthonormal system $\{\overline{\psi}_\iota\}_{\iota=1}^{\infty}$ of $W_2^{(3,2)}(F)$:

$$\overline{\psi}_\iota(\zeta,\eta) = \sum_{k=1}^{\iota} \varkappa_{\iota k}\psi_k(\zeta,\eta), \quad \varkappa_{\iota\iota} > 0,\ \iota = 1,2,\ldots, \tag{9}$$

with the following cases of the orthogonalization coefficients $\varkappa_{\iota k}$:

$$\begin{cases} \varkappa_{\iota\ell} = \frac{1}{\|\psi_1\|} & \text{for} \quad \iota = \ell = 1, \\ \varkappa_{\iota\ell} = \frac{1}{e_\iota} & \text{for} \quad \iota = \ell \neq 1, \\ \varkappa_{\iota\ell} = -\frac{1}{e_\iota}\sum_{k=\ell}^{\iota-1} C_{\iota k}\varkappa_{k\ell} & \text{for} \quad \iota > \ell, \end{cases}$$

where $e_\iota = \sqrt{\|\psi_\iota\|^2 - \sum_{k=1}^{\iota-1} C_{\iota k}^2}, C_{\iota k} = \langle \psi_\iota, \overline{\psi}_k\rangle_{W_2^{(3,2)}}$, and $\{\psi_\iota\}_{\iota=1}^{\infty}$ form an orthonormal system in the space $W_2^{(3,2)}(F)$.

Remark 3.1: It is easy to infer the following formula

$$\psi_\iota(\zeta,\eta) = \Im_{(\kappa,\nu)}Q_{(\zeta,\eta)}(\kappa,\nu)|_{(\kappa,\nu)=(\zeta_\iota,\eta_\iota)}$$

where the operator $\Im_{(\kappa,\nu)}$ indicates that the operator $\Im$ is applied to the variables κ and ν.

Theorem 3.1: $\psi_\iota(\zeta,\eta) \in W_2^{(3,2)}(F)$.

Theorem 3.2: Let the set $\{\zeta_\iota,\eta_\iota\}_{\iota=1}^{\infty}$ be dense on F, then we say that $\{\psi_\iota\}_{\iota=1}^{\infty}$ is a complete system of the space $W_2^{(3,2)}(F)$.

Proof: We have first

$$\begin{aligned}\psi_\iota(\zeta,\eta) = (\Im^*\wp_\iota)(\zeta,\eta) &= \left\langle (\Im^*\wp_\iota)(\kappa,\nu), Q_{(\kappa,\nu)}(\zeta,\eta)\right\rangle_{W_2^{(3,2)}} \\ &= \left\langle (\wp_\iota)(\kappa,\nu), \Im_{(\kappa,\nu)}Q_{(\kappa,\nu)}(\zeta,\eta)\right\rangle_{W_2^{(1,1)}} \\ &= \Im_{(\kappa,\nu)}Q_{(\zeta,\eta)}(\kappa,\nu)|_{(\kappa,\nu)=(\zeta_\iota,\eta_\iota)}\end{aligned}$$

Let us secondly put

$$\langle \vartheta(\zeta,\eta), \psi_\iota(\zeta,\eta)\rangle_{W_2^{(3,2)}} = 0,\ \iota = 1,2,\ldots,$$

for each fixed $\vartheta(\zeta,\eta) \in W_2^{(3,2)}(F)$.
So

$$\begin{aligned}\langle \vartheta(\zeta,\eta), (\Im^*\wp_\iota)(\zeta,\eta)\rangle_{W_2^{(3,2)}} &= \langle \Im\vartheta(\zeta,\eta), \wp_\iota(\zeta,\eta)\rangle_{W_2^{(1,1)}} \\ &= (\Im\vartheta)(\zeta_\iota,\eta_\iota) = 0.\end{aligned}$$

and note that $\{(\zeta_\iota,\eta_\iota)\}_{\iota=1}^{\infty}$ is a dense set in F, so $\Im\vartheta(\zeta,\eta) = 0$. Because of the existence of $\Im^{-1}$, $\vartheta(\zeta,\eta) = 0$. ■

Theorem 3.3: Take $\{(\zeta_\iota,\eta_\iota)\}_{\iota=1}^{\infty}$ is a dense set on F and assume the solution of (1) is unique on $W_2^{(3,2)}(F)$, then the solution of the problem (8) is given by

$$\vartheta(\zeta,\eta) = \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}G(\zeta_k,\eta_k,\vartheta(\zeta_k,\eta_k),\partial_{\eta\zeta\zeta}\vartheta(\zeta_k,\eta_k))\overline{\psi}_\iota(\zeta,\eta). \tag{10}$$

Proof: We drive from the completeness of $\left\{\bar{\psi}_\iota(\zeta,\eta)\right\}_{\iota=1}^{\infty}$ in $W_2^{(3,2)}(F)$ that

$$\begin{aligned}\vartheta(\zeta,\eta) &= \sum_{\iota=1}^{\infty}\left\langle \vartheta(\zeta,\eta), \bar{\psi}_\iota(\zeta,\eta)\right\rangle_{W_2^{(3,2)}}\bar{\psi}_\iota(\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}\left\langle \vartheta(\zeta,\eta), \psi_k(\zeta,\eta)\right\rangle_{W_2^{(3,2)}}\bar{\psi}_\iota(\zeta,\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}\left\langle \vartheta(\zeta,\eta), \Im^*\wp_k(\zeta,\eta)\right\rangle_{W_2^{(3,2)}}\bar{\psi}_\iota(\zeta,\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}\left\langle \Im\vartheta(\zeta,\eta), \wp_k(\zeta,\eta)\right\rangle_{W_2^{(1,1)}}\bar{\psi}_\iota(\zeta,\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}\left\langle \Im\vartheta(\zeta,\eta), \tilde{Q}_{(\zeta,\eta)}(\zeta_k,\eta_k)\right\rangle_{W_2^{(1,1)}}\bar{\psi}_\iota(\zeta,\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}\Im\vartheta(\zeta_k,\eta_k)\bar{\psi}_\iota(\zeta,\eta) \\ &= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota} \varkappa_{\iota k}G(\zeta_k,\eta_k,\vartheta(\zeta_k,\eta_k),\partial_{\eta\zeta\zeta}\vartheta(\zeta_k,\eta_k))\bar{\psi}_\iota(\zeta,\eta).\end{aligned}$$

■

Now we can obtain the approximate solution $\vartheta_n(\zeta,\eta)$ by takin finite number of terms in (10) as follows

$$\vartheta_n(\zeta,\eta) = \sum_{\iota=1}^{n}\sum_{k=1}^{\iota} \varkappa_{\iota k}G(\zeta_k,\eta_k,\vartheta(\zeta_k,\eta_k),\partial_{\eta\zeta\zeta}\vartheta(\zeta_k,\eta_k))\overline{\psi}_\iota(\zeta,\eta). \tag{11}$$

Lemma 3.1: If $G(\zeta,\eta,\vartheta(\zeta,\eta),\partial_{\eta\zeta\zeta}\vartheta(\zeta,\eta))$ is continuous and $\vartheta_n \to \hat{\vartheta}$ for $(\zeta_n,\eta_n) \to (\kappa,\nu)$, then

$$G(\zeta_n,\eta_n,\vartheta_{n-1}(\zeta_n,\eta_n),\partial_{\eta\zeta\zeta}\vartheta_{n-1}(\zeta_n,\eta_n)) \to G(\kappa,\nu,\hat{\vartheta}(\kappa,\nu),\partial_{\eta\zeta\zeta}\hat{\vartheta}(\kappa,\nu))$$

Theorem 3.4: Suppose that $\|\vartheta_n\|$ is a bounded in (8) and the solution of (11) is unique. If $\{(\zeta_\iota,\eta_\iota)\}_{\iota=1}^{\infty}$ is a dense set in F, so the approximate solution $\vartheta_n(\zeta,\eta)$ converges to $\vartheta(\zeta,\eta)$ of (8) and

$$\vartheta(\zeta,\eta) = \sum_{\iota=1}^{\infty}\mathfrak{A}_\iota\overline{\psi}_\iota(\zeta,\eta) \tag{12}$$

Proof: We will first demonstrate that $\vartheta_n(\zeta,\eta)$ is convergente. We deduce from (11)

$$\vartheta_{n+1}(\zeta,\eta) = \vartheta_n(\zeta,\eta) + \mathfrak{A}_{n+1}\overline{\psi}_\iota(\zeta,\eta). \tag{13}$$

From the orthonormality of $\{\overline{\psi}_\iota\}_{\iota=1}^{\infty}$, we provide that

$$\|\vartheta_{n+1}\|^2 = \|\vartheta_n\|^2 + \mathfrak{A}_{n+1}^2 = \sum_{\iota=1}^{n+1}\mathfrak{A}_\iota^2 \tag{14}$$

So, $\|\vartheta_{n+1}\| > \|\vartheta_n\|$.
Because of the boundedness of $\|\vartheta_n\|$, we deduce that $\|\vartheta_n\|$ is convergent and $\exists\ M$ such that

$$\sum_{\iota=1}^{\infty}\mathfrak{A}_\iota^2 = M. \tag{15}$$

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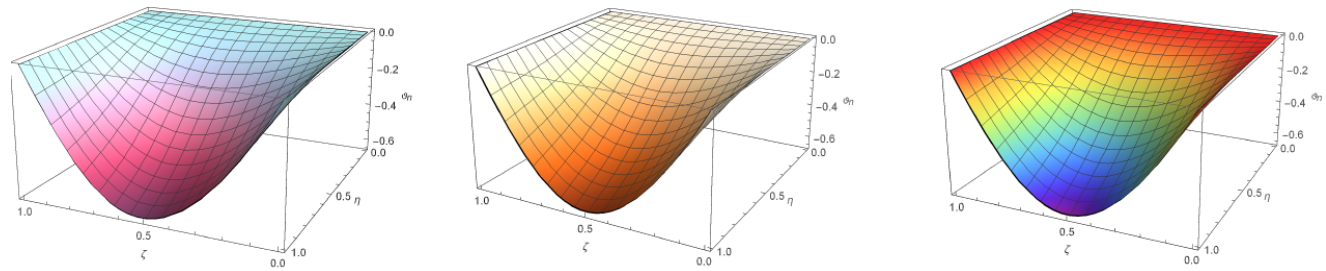

Fig. 1. Approximate solutions of Example 1 for different grid points and various values of $\alpha, \alpha = 1, \alpha = 0.85$ and $\alpha = 0.6$ respectively from left to right.

So, this indicates that $\{\mathfrak{A}_\iota\}_{\iota=1}^{\infty} \in l^2$. if $p > n$, then

$$\begin{aligned}\|\vartheta_p - v_n\|^2 &= \|\vartheta_p - \vartheta_{p-1} + \vartheta_{p-1} - \cdots + \vartheta_{n+1} - \vartheta_n\|^2 \\ &= \|\vartheta_p - \vartheta_{p-1}\|^2 + \cdots + \|\vartheta_{n+1} - \vartheta_n\|^2 \quad (16)\end{aligned}$$

On account of

$$\|\vartheta_p - \vartheta_{p-1}\|^2 = \mathfrak{A}_p^2 \quad (17)$$

Hence,

$$\|\vartheta_p - \vartheta_n\|^2 = \sum_{\iota=n+1}^{p} B_l^2 \to 0, \text{ as } n \to \infty \quad (18)$$

Thus, $\vartheta_n \to \hat{\vartheta}$ as $n \to \infty$ by the completeness of $W_2^{(3,2)}(D)$.

Next, we will demonstrate that $\hat{\vartheta}$ is the representation solution of (8).
We take limits in (11), we obtain

$$\hat{\vartheta}(\zeta, \eta) = \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \overline{\psi}_\iota(\zeta, \eta). \quad (19)$$

Note that

$$\left(\Im\hat{\vartheta}\right)(\zeta, \eta) = \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \Im\overline{\psi}_\iota(\zeta, \eta). \quad (20)$$

$$\begin{aligned}\left(\Im\hat{\vartheta}\right)(\zeta_l, \eta_l) &= \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \Im\overline{\psi}_\iota(\zeta_l, \eta_l) \quad (21)\\ &= \sum_{\iota=1}^{\infty} \mathfrak{A}_i \left\langle \Im\overline{\psi}_\iota(\zeta, \eta), \wp_l(\zeta, \eta)\right\rangle_{W_2^{(1,1)}} \\ &= \sum_{\iota=1}^{\infty} \mathfrak{A}_i \left\langle \overline{\psi}_\iota(\zeta, \eta), \Im^* \wp_l(\zeta, \eta)\right\rangle_{W_2^{(3,2)}} \quad (22)\\ &= \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \left\langle \overline{\psi}_\iota(\zeta, \eta), \psi_l(\zeta, \eta)\right\rangle_{W_2^{(3,2)}} \quad (23)\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{l=1}^{\iota} \varkappa_{\iota l} \left(\Im\hat{\vartheta}\right)(\zeta_l, \eta_l) &= \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \left\langle \overline{\psi}_\iota(\zeta, \eta), \sum_{l=1}^{\iota} \varkappa_{\iota l} \psi_l(\zeta, \eta)\right\rangle_{W_2^{(3,2)}} \\ &= \sum_{\iota=1}^{\infty} \mathfrak{A}_\iota \left\langle \overline{\psi}_\iota(\zeta, \eta), \overline{\psi}_l(\zeta, \eta)\right\rangle_{W_2^{(3,2)}} = \mathfrak{A}_l \quad (24)\end{aligned}$$

We have

$$\Im\hat{\vartheta}(\zeta_l, \eta_l) = G(\zeta_l, \eta_l, \vartheta_{l-1}(\zeta_l, \eta_l), \partial_{\eta\zeta\zeta}\vartheta_{l-1}(\zeta_l, \eta_l)) \quad (25)$$

Since $\{(\zeta_\iota, \eta_\iota)\}_{\iota=1}^{\infty}$ is dense on D, for each $(\zeta, \eta) \in D$, there exists a subsequence $\{(\zeta_{nj}, \eta_{nj})\}_{j=1}^{\infty}$ such that $(\zeta_{nj}, \eta_{nj}) \to (\zeta, \eta), j \to \infty$.
Noting that

$$\Im\hat{\vartheta}(\zeta_{nj}, \eta_{nj}) = G(\zeta_{nj}, \eta_{nj}, \vartheta_{nj-1}(\zeta_{nj}, \eta_{nj}), \partial_{\eta\zeta\zeta}\vartheta_{nj-1}(\zeta_{nj}, \eta_{nj})). \quad (26)$$

Let $j \to \infty$, by the continuity of G and Lemma 3.1, we can write

$$\Im\hat{\vartheta}(\zeta, \eta) = G(\zeta, \eta, \vartheta(\zeta, \eta), \partial_{\eta\zeta\zeta}\vartheta(\zeta, \eta)). \quad (27)$$

which demonstrates that $\hat{\vartheta}(\zeta, \eta)$ provide (8).

■

IV. Numerical applications

We present in this section some numerical results where the RKHS approach is applied for a nonlinear partial differential equation with time-fractional order.

Example 1: Let us consider the nonlinear time fractional order partial differential equation :

$$\begin{cases} \mathfrak{D}_\eta^\alpha \vartheta - \vartheta\vartheta_\zeta - \vartheta_{\zeta\zeta} = \hbar(\zeta, \eta), \ 0 < \zeta, \eta < 1, \ 0 < \alpha \le 1, \\ \vartheta(\zeta, 0) = 0, \quad 0 \le \zeta \le 1, \\ \vartheta(0, \eta) = \vartheta(1, \eta) = 0 \quad 0 \le \eta \le 1. \end{cases} \quad (28)$$

where

$$\hbar(\zeta, \eta) = -\pi^2\eta^{2\alpha}\sin(\pi\zeta) - \pi\eta^{4\alpha}\cos(\pi\zeta)\sin(\pi\zeta) - \frac{2\alpha\eta^\alpha\Gamma(2\alpha)\sin(\pi\zeta)}{\Gamma(1+\alpha)},$$

and $\vartheta(\zeta, \eta) = \cos(\frac{\pi}{2} + \pi\zeta)\eta^{2\alpha}$ is the exact solution of (28). We obtained the results wich are presented in Fig. 1 by using RKHS method. The grid points are taken for $n = 16$. Fig. 1 show the graphics of the approximate solutions $\vartheta_n(\zeta, \eta)$ for $\alpha = 1, \alpha = 0.85$ and $\alpha = 6$ respectively.

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