N^0 :...... / Faculty / UMBB / 2025

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

M'HAMED BOUGARA UNIVERSITY OF BOUMERDES

Faculty of Sciences - Department of Mathematics

DOCTORAL THESIS

Presented by: **BARRY Kalifa Lassana**

Major: **Mathematics**

Sub major: **Partial differential equations**

STABILIZATION OF DIFFERENT EVOLUTION PROBLEMS BY INTERNAL DYNAMIC CONTROLLERS

Defense Date: April 21, 2026

Committee members:

ADJABI Yacine	Professor	U. M. B. B.	Chief examiner
BENAKMOUM Saliha	Associate Professor	U. M. B. B.	Examiner
GRAZEM Mohamed	Associate Professor	U. M. B. B.	Examiner
BOUDJERADA Rachida	Associate Professor	U. S. T. H. B.	Examiner
LAOUBI Karima	Associate Professor	U. M. B. B.	Advisor
NOUR Abdelkader	Emeritus Professor	U. M. B. B.	Guest.

Academic year 2025/2026

N^0 :...... / Faculté / UMBB / 2025

RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ M'HAMED BOUGARA DE BOUMERDÈS

Faculté des sciences - Département de Mathématiques

THÈSE

pour l'obtention du diplôme de Doctorat en Mathématiques 3ème cycle LMD

Présentée par: **BARRY Kalifa Lassana**

Filière: **Mathématiques**

Spécialité: **Équations aux Dérivées Partielles (E. D. P)**

STABILISATION DE DIFFÉRENTS PROBLÈMES D'ÉVOLUTION PAR DES CONTRÔLEURS INTERNES ET DYNAMIQUES

Soutenu le: 21 Avril 2026 devant le jury composé de:

Mr. ADJABI Yacine	Professeur	U. M. B. B.	Président du jury
Mme BENAKMOUM Saliha	Maître de Conférences/A	U. M. B. B.	Examinatrice
Mr. GRAZEM Mohamed	Maître de Conférences/A	U. M. B. B.	Examinatrice
Mme BOUDJERADA Rachida	Maître de Conférences/A	U. S. T. H. B.	Examinatrice
Mme LAOUBI Karima	Maître de Conférences/A	U. M. B. B.	Directrice de thèse
Mr. NOUR Abdelkader	Professeur Émérite	U. M. B. B.	Invité.

Année académique 2025/2026

Acknowledgments

I dedicate this work to my dear parents (rest in peace), my brothers and sisters and my uncles and aunts. Their unconditional support and encouragement have been a great help.

I am profoundly grateful to my advisor, Professor **LAOUBI Karima**, for her excellent guidance, her wise advice, and her unwavering patience throughout this thesis. I am also grateful for her availability, her support, and her unwavering confidence in me. I am so grateful that words cannot adequately express how much I appreciate it.

I thank Professor **ADJABI Yacine** for giving me the immense honor of chairing the judging committee.

I thank Associate Professor **BENAKMOUM Saliha**, Associate Professor **GRAZEM Mohamed**, Associate Professor **BOUDJERADA Rachida** and Professor **NOUR Abdelkader** for agreeing to review this work. Their presence in itself is a great honor.

By the way, I am grateful to my employer, Department of Mathematics, M'Hamed Bougara University of Boumerdes (UMBB), for training me throughout my LMD program (Bachelor-Master-Doctorate).

Finally, I say thank you to all my friends and colleagues for their moral support and for knowledge exchange, during my research. The road was tough, but here we are.

BARRY Kalifa Lassana

Boumerdes, August 2025

Abstract

This thesis explores broadly the stabilization of complex dynamic systems by means of internal controllers. Specially, the asymptotic behavior of damped wave ζ in an annular domain $\Omega \subset \mathbb{R}^n, n \geq 2$, bounded by two edges Γ_0 and Γ_1 of class C^2. The inner edge Γ_0 is fasten so as to prevent any movement on this edge (Homogenous Dirichlet condition):

$$\zeta_{|\Gamma_0} = 0.$$

As for the outer edge, the condition typically considered is:

$$\alpha(x)\zeta_{tt} + \partial_\nu \zeta - \Delta_T \zeta = 0, \quad \text{over } \Gamma_1 \times \mathbb{R}_+, \quad \alpha \in L^\infty(\Gamma_1)$$

where ζ_{tt} represents the acceleration of ζ (presence of kinetic energy),
$\alpha \geq 0$ parameter allowing the presence\or absence of kinetic energy on Γ_1,
$\nu \in \mathbb{R}^n$ normal vector pointing out of Ω along Γ_1,
Δ_T Laplace-Beltrami operator.

First of all, we start by the case where $\alpha > 0$. In second position, we analyze the case where $\alpha = 0$.

For each case, we firstly demonstrate the existence and uniqueness of the solution by the semigroup method and the stability of each solution by analyzing the released energy over time. This energy analysis reveals a decrease in energy over time, thus suggesting a possible strong stability.

For $\alpha > 0$, we show the strong stability of the solution by exploiting the Arendt-Batty theorem combined with a unique continuation result. Then, we show the exponential stability for $\alpha > 0$ (resp. $\alpha = 0$) by a frequency approach of the Ω domain (resp. the Nakao method).

Finally, we conduct a series of numerical simulations to illustrate this exponential stability.

Keywords & Phrases: Carleman estimate; Exact controllability; Finite differences method; Geometric Condition; HUM method; Kinetic boundary conditions; Multiplier method; Nakao's criterion; Numerical simulation; Stabilization; Damped Wave equation; Wentzell boundary conditions.

Abstract (French version)

Cette thèse explore au large la stabilisation de systèmes dynamiques complexes à travers des contrôleurs internes. Elle analyse plus particulièrement le comportement asymptotique de l'onde ζ amortie à l'intérieur d'un domaine annulaire $\Omega \subset \mathbb{R}^n, n \geq 2$, délimité par ses deux bords $\Gamma_0, \Gamma_1 \in C^2$. Le bord interne Γ_0 est fixée de sorte à empêcher tout mouvement sur ce bord (condition Dirichlet homogène) :

$$\zeta_{|\Gamma_0} = 0.$$

Quant au bord externe Γ_1, la condition typiquement considérée est:

$$\alpha(x)\zeta_{tt} + \partial_\nu \zeta - \Delta_T \zeta = 0, \quad \text{sur } \Gamma_1 \times \mathbb{R}_+, \quad \alpha \in L^\infty(\Gamma_1),$$

où ζ_{tt} représente l'accélération de l'onde ζ (présence d'énergie cinétique),
$\alpha \geq 0$ paramètre permettant la présence\ou l'absence d'énergie cinétique sur Γ_1,
$\nu \in \mathbb{R}^n$ vecteur normal dirigé vers l'extérieur du domaine Ω le long de Γ_1,
Δ_T opérateur de Laplace-Beltrami.

Tout d'abord, on commence par étudier le cas où $\alpha > 0$ puis on analyse le cas $\alpha = 0$. Dans chaque cas, nous montrons d'abord l'existence et l'unicité de la solution par la méthode des sémi-groupes et la stabilité de chaque solution par une analyse de l'énergie qu'elle libère aucours du temps. Cette analyse de l'énergie révèle une décroissance en énergie au cours du temps, suggérant ainsi une possible stabilité forte.

Pour $\alpha > 0$, nous montrons la stabilité forte de la solution en exploitant le théorème d'Arendt-Batty combiné à un resultat d'unique continuation. Ensuite, nous montrons la stabilité exponentielle pour $\alpha > 0$ (resp. $\alpha = 0$) par une approche frequentielle du domaine Ω (resp. la méthode de Nakao).

Enfin, nous ménons une série de simulations numériques, pour illustrer cette stabilité exponentielle.

Mot-clés & Expressions: Condition Cinétique; Condition de Wentzell; Condition Géométrique; Contrôlabilité exacte; Différences Finies; Estimation de Carleman; Équation d'Onde amortie; Méthode de HUM; Méthode de Multiplicateur; Méthode de Nakao; Stabilisation.

Abstract (Arabic version)

تستكشف هذه الأطروحة بشكل موسع استقرار الأنظمة الديناميكية المعقدة باستخدام وحدات تحكم داخلية. وتحديدًا، السلوك المقارب للموجة المخمدة ζ في نطاق حلقي $\Omega \subset \mathbb{R}^n, n \geq 2$، محاطًا بحافتين Γ_0 و Γ_1 من الفئة C^2. الحافة الداخلية Γ_0 مثبتة لمنع أي حركة عليها (شرط ديريتشليت المتجانس):

$$\zeta_{|\Gamma_0} = 0.$$

أما بالنسبة للحافة الخارجية، فالشرط المُعتمد عادةً هو:

$$\alpha(x)\zeta_{tt} + \partial_\nu \zeta - \Delta_T \zeta = 0, \quad \text{على } \Gamma_1 \times \mathbb{R}_+, \quad \alpha \in L^\infty(\Gamma_1)$$

حيث يمثل ζ_{tt} تسارع ζ (وجود طاقة حركية)

$\alpha \geq 0$ معامل يسمح بوجود أو غياب طاقة حركية على Γ_1,

$\nu \in \mathbb{R}^n$ متجه عمودي يشير إلى خارج Ω على طول Γ_1,

Δ_T عامل لابلاس-بيلترامي.

أولاً، نبدأ بالحالة التي يكون فيها $\alpha > 0$. ثانياً، نحلل الحالة التي يكون فيها $\alpha = 0$.

في كل حالة، نوضح أولاً وجود الحل وفرادته باستخدام طريقة شبه المجموعة، ونوضح استقرار كل حل من خلال تحليل الطاقة المنبعثة منه مع مرور الوقت. يكشف تحليل الطاقة هذا عن انخفاض في الطاقة مع مرور الوقت، مما يشير إلى احتمال وجود استقرار قوي.

بالنسبة إلى $\alpha > 0$، نوضح الاستقرار القوي للحل من خلال استخدام نظرية أرندت-باتي مع نتيجة استمرارية فريدة.
ثم نوضح الاستقرار الأسي لـ $\alpha > 0$ (أو $\alpha = 0$) باستخدام نهج ترددي للمجال Ω (أو طريقة ناكاو).
وأخيراً، نجري سلسلة من عمليات المحاكاة العددية لتوضيح هذا الاستقرار الأسي.

List of Abbreviations & Notations

In order to have a more readable and concise text, we will use the following acronyms and notations to be used when there is no specification:

WE: Wave Equation

HWE: Hyperbolic Wave Equation

PDE: Partial Differential Equation

ζ an arbitrary wave.

Ω subset of $\mathbb{R}^n$ bounded by $\Gamma = \partial\Omega$.

Γ smooth boundary of Ω, such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

ω subset of Ω, is the damped region.

$\chi_\omega = 1$ in ω and 0 otherwise, is the characteristic function over Ω.

$int(\Omega)$ interior of Ω.

$\overline{\Omega} = int(\Omega) \cup \Gamma$ closure of Ω.

$dim(\Omega)$ dimension of Ω.

$^\top$ transposed symbol of vector.

$\nu = (\nu_1, \dots, \nu_n)^\top \in \mathbb{R}^n$, normal boundary vector pointing out of Ω.

∂_ν derivative in outward normal direction.

$\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^\top$ gradient operator.

$\nabla_T = (\frac{\partial_T}{\partial x_1}, \dots, \frac{\partial_T}{\partial x_n})^\top$ tangential gradient operator.

$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ Laplacian.

$\vec{\Delta}\vec{V} = (\Delta V^x, \Delta V^y, \Delta V^z)^\top$ is the vector laplacian of a vector field $V = (V^x, V^y, V^z)$.

$\Delta_T = \sum_{i=1}^n \frac{\partial_T^2}{\partial x_i^2}$ Laplace-Beltrami operator.

$C^k(\Omega)$ space of k-times continuously differentiable functions over Ω.

$\mathscr{D}(\Omega)$ space of test functions.

$\mathscr{L}(E)$ space of bounded linear maps on E.

$\mathbb{N}^*$ nonzero positive integers.

$\mathbb{R}^*$ nonzero real numbers.

$*$ dual mark except for the space $\mathbb{N}^*, \mathbb{R}^*$.

X^* dual space of X.

X^{**} bi-dual space of X.

L^p Lebesgue space on Ω.

$W^{m,p}$ m-order Sobolev space in L^p.

H^m m-order Sobolev space in $L^2(\Omega)$.

$\rightharpoonup$ convergence in weak topology.

$\overset{*}{\rightharpoonup}$ convergence in weak-* topology.

$\hookrightarrow$ continuous embedding.

$\hookrightarrow\hookrightarrow$ compact embedding.

$ess\ supA$ essential supremum of the set A.

$\mathbb{C}$ set of complex numbers.

$\Re e(z)$ real part of $z \in \mathbb{C}$.

$\bar{z}$ complex conjugate of $z \in \mathbb{C}$.

Contents

1

General Introduction

1.1 Introduction

Mathematical modeling of vibratory phenomenon in continuous medium is very important in many scientific and technical fields, especially, in structural mechanics, acoustic, or in the conception of smart materials. In this context, the wave equations constitute a fundamental tool for describing the disturbance propagation with or without dissipation. The use of internals damping or dynamic boundary conditions allow to establish a result of energy decay for the system, which is crucial when a rapid come back of the medium to the rest state is needed.

In this context, this work explores successively three formulations related to wave problem in annular medium $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and taking into account various dissipation mechanisms.

- **The first model** consider the wave equation with linear internal damping localized in Ω and mixed boundary conditions (Dirichlet-Neumann). The main goal for this one is, to analyse the well-posedness of the system and the outcomes from the result of energy decay over time. The considered wave is as follows:

$$\begin{cases} \theta_{tt}(x,t) - \Delta\theta(x,t) + d(x)\theta_t(x,t) = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \theta = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}_+, \\ m(x)\theta_{tt}(x,t) - \Delta_T\theta(x,t) + \partial_\nu\theta = 0, & (x,t) \in \Gamma_1 \times \mathbb{R}_+, \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad x \in \Omega, \quad \theta_t(x,0) = \theta_2(x), & x \in \Gamma_1 \end{cases} \tag{1.1}$$

  where for a certain $d_0 > 0$, $d(x) = d_0\chi_\omega(x)$, $x \in \Omega$ allows internal damping; and χ_ω is the characteristic function over Ω;
  $0 < m \in L^\infty(\Gamma_1)$ measure the mass density at each point $x \in \Gamma_1$.

- **The second model** introduces a restricted dynamic control over the outer edge of Ω via modified Wentzell-type condition, while holding the internal damping. This configuration allows a greater flexibility in the treatment of border effects, in particular through the use of Laplace-Beltrami operator. The analysis focuses on the exponential stability of the system using the method of Nakao.

  The model considered in this case is as follows:

$$\begin{cases} \theta_{tt}(x,t) - \Delta\theta(x,t) + d(x)\theta_t(x,t) = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \theta = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}_+, \\ \partial_\nu \theta = \Delta_T \theta(x,t), & (x,t) \in \Gamma_1 \times \mathbb{R}_+, \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

- **The third model** is inspired by the previous ones, in particular, by the system (1.1) and the stabilization method of the second model (Method of Nakao). The third model deepens the numerical study of exponential stabilization using a finite difference scheme applied to a one-dimensional reduction of the problem. This approach allows us to verify the validity of the theoretical results by simulations, highlighting the influence of the dissipation coefficient d_0 and the mass density m on the energy decay rate.

In order to understand the physical interpretation of the boundary conditions, we advise readers to read the references [34, 63] and those cited therein.

As we can see, the three models consider the same internally damped wave but which differ either on the presence of kinetic energy on Γ_1 (term $m\vartheta_{tt}$), or on the stabilization method. Recently, many searchers have been interested into asymptotic behavior of such a wave [2, 17, 20, 31, 46, 53, 55, 63, 74, 88, 91]. However, the study of weak stabilization such as logarithmic and polynomial of these models is not the aim of this thesis. For these cases, we invite readers to see [16, 31–33, 53, 55, 68, 79, 97] and the references cited therein.

In 2009, the authors of [17] demonstrated the exponential stability of a model where (1.2) is a particular case in term of linear damping:

$$\begin{cases} \theta_{tt} - \Delta\theta + a(x)g(\theta_t) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_\nu \theta - \Delta_T \theta = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ \theta = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ (\theta(x,0), \theta_t(x,0)) = (\theta_0(x), \theta_1(x)), & \text{in } \Omega, \end{cases} \tag{1.3}$$

with $a = a(x) \geq a_0 > 0$ on $\omega = \omega_0 \cup \omega_1$, ω_i neighborhood of Γ_i in Ω, and $a = 0$ otherwise, g represents the monotone increasing nonlinear damping and satisfies for two constants $k, K > 0$:

$$k|s| \leq g(s) \leq K|s|, \quad \text{for all } |s| \geq 1.$$

The authors used a method that is quite common in the nonlinear case, introduced by I. Lasiecka and D. Tataru [56] in order to reduce the stabilization conditions and where damping acts on the boundary. This method combines the multiplier method and cut-off functions inspired by J. Lions [65]. The authors even affirm that the technique used remains valid when Γ_1 is a thin-layer slightly rigid and keeps the following dynamic:

$$\theta_{tt} - \Delta_T \theta + \partial_\nu \theta = 0, \quad \text{on } \Gamma_1 \times (0, +\infty).$$

However, the only problem with this technique for our model is that it contains a multitude of conditions we need to exploit, to determine the influence of internal damping (coefficient d_0) along m (weight of Γ_1) and conversely, the influence of m along d_0.

The study carried out on the second and third models in this thesis, considerably reduces these conditions by the use of Nakao's method and numerical simulations.

And this, allows to define an acceptable choice of d_0 and m which leads to an exponential comeback of the wave to the rest state.

In the same year, A. Münch [74] study the impact of the damped region ω and the coefficient of dissipation of the following model:

$$\begin{cases} y''_{\omega,a} - \Delta y_{\omega,a} + a(x) y'_{\omega,a} = 0, & \text{dans } \Omega \times \mathbb{R}_+, \\ y_{\omega,a} = 0, & \text{sur } \Gamma \times \mathbb{R}_+, \\ y_{\omega,a}(x,0) = y_0(x), \quad y'_{\omega,a}(x,0) = y_1(x), & \text{dans } \Omega \end{cases} \tag{1.4}$$

where $y_{\omega,a} = y_{\omega,a}(x,t)$ represents the wave damped in the region ω, and a is the coefficient of dissipation such that:

$$a(x) = a\chi_\omega(x), \quad x \in \Omega.$$

The main goal was to find $\omega \subset \Omega$ (*resp.* $a > 0$) which considerably minimize the released energy $E(\omega, a, t)$ by the system. And that, is to study the well-posedness of the following optimizations problem:

$$(P_\omega): \inf_{\omega \subset \Omega} E(\omega, a, t), \quad a \text{ given}.$$

$$(P_a): \inf_{a > 0} E(\omega, a, t), \quad \omega \text{ given}.$$

A. Münch found that (P_ω) is well-posed (*resp.* ill-posed) when a is small enough (*resp.* excessively large).

A similar result is obtained by the application of Nakao's method in this thesis.

Recently in [63], a more general system has been studied. Specially, with the variation in time of the dissipation coefficient a and the presence of source function:

$$\begin{cases} u_{tt} - \Delta u + \gamma(t)a(x)g_0(u_t) + f(u) = 0, & \text{dans } \Omega \times \mathbb{R}_+, \\ u_{tt} + \partial_\nu u - \Delta_T u + g_1(u_t) = 0, & \text{sur } \Gamma_1 \times \mathbb{R}_+, \\ u = 0, & \text{sur } \Gamma_0 \times \mathbb{R}_+, \end{cases} \tag{1.5}$$

In physical perspective, this system could model the vibrations of a thin metal membrane with a hole in its center where it is stably clamped ($u_{|\Gamma_0} = 0,\ \Gamma_0 \neq \emptyset$). In addition, a damper $a(x)g(u_t)$ which strength varies in time (thanks to $\gamma(t)$) is placed over a portion of its surface along the outer boundary Γ_1 where act some nonlinear friction forces $g_1(u_t)$.

The authors used the same combination of the methods introduced by [49, 56, 65] to demonstrate the exponential stability of the system (1.5).

The main goal of this thesis, is to establish the exponential stability of both models (1.1) and (1.2) by using an approach analyzing it, according d_0 and/or m.

1.2 Historical Overview: From the Founder to the Present

This section provides a chronological report of facts which leads to the use of different control mechanisms on the hyperbolic wave. We will claim that this historical report is exhaustive.

According to C. G. Douglas, until the 17th century, the concept of wave equation did not exist. The only known things about waves, were its charateristics: wavelength, wave speed, wave frequency (see [26, p. 18]).

1.2.1 Emergence of 2^{nd} order HWE

In 1743, the scientist Jean Le Rond d'Alembert (AD 1717- AD 1783) stated in his book entitled "**Traité de Dynamique**", the fundamental classical laws of motion later called "**Principle of**

D'Alembert" (see [8, 85] or site[1]). For the first time in history, this principle allows him to model a vibrating string of length l and fixed at its ends:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial s^2}, & (t,s) \in \mathbb{R}_+ \times [0;l] \\ y(t,0) = y(t,l) = 0, & t \in \mathbb{R}_+ \end{cases} \tag{1.6}$$

where $y = y(t,s)$ is the position of the string (wave position),
s abscissa representing a point or portion of the string,
t denotes time.

In 1746, D'Alembert solved (1.6) by a variant of separation of variables method early introduced by Euler in 1730 (see [29, p. 30-34]). Thus making this method, the first efficient method in solving hyperbolic second-order PDE. It was the advent of a whole new era.

After these results of D'Alembert, many questions remained unanswered. Especially, the mathematical solution $y = y(t,s)$ established by D'Alembert:

$$y(t,s) = \psi(t+s) + \Gamma(t-s), \quad \psi, \Gamma \text{ arbitrary functions},$$

and its consistency with the physical solution already known at this epoch. The physical approach would like that the shape of a vibrating string is a sinusoid (as sine curve). Thus, the physicists of this time, notably Daniel Bernoulli, combined various simple vibrations (arcs) represented by

$$\alpha sin\left(\frac{2k\pi}{l}\right), k \in \mathbb{N}^*,$$

where α is a constant depending on the simple vibration amplitude.

We had to wait until 1822 that Fourier introduced the series known as "**Fourier Series**" for an agreement between both approaches.

The main goal and motivation of D'Alembert was to study challenging physical phenomena from a mathematical perspective. Which also led to a lively discord between the three scientists: D'Alembert, Euler, Bernoulli.

1.2.2 Emergence of another 2^{nd} order HWE

By the first half of 19th century, several laws had already been established regarding magnetism and electricity. However, it was in 1864 that James Clerk Maxwell unified these laws into a

[1] http://cm2.ens.fr/content/le-traite-de-dynamique-de-dalembert-2697

series of four equations later called "**Maxwell's equations**". A combination of these equations in vacuum gave rise to the electromagnetic WE in vacuum:

$$\vec{\Delta}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}, \quad \vec{\Delta}\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

where $\vec{E}$ (resp. $\vec{B}$) denotes electric (resp. magnetic) vector field which propagates in vacuum,
$\vec{\Delta}$ vector Laplacian,
c constant depending on magnetic and electric constants.

As we can see, these equations generalize in term of spatial dimension that of the string given in $(1.6)_1$.

1.2.3 Consideration of damping term $\frac{\partial \zeta}{\partial t}$

After our research, we cannot say exactly when searchers have considered for the first time, a damping term in HWE. And for good reason, some scientists have purposely studied this concept (damped HWE) while others have just taken it as an example.

In 1870, the journal *Bartholomew Price* published a book entitled "**Acoustics**". This book summarize the works of the astro-mathematician William Fishburn Donkin (1814-1869) before his passing. In the book, the damping phenomenon is explained as a retarding force proportional to the wave speed (see [25, p. 110]). And he models the presence of such a force acting on a string stretched then slightly pulled apart and released, by:

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} = a^2\frac{d^2y}{dx^2}$$

where $y = y(x,t)$ is the position of the string,
$a \neq 0, k > 0$ constants.

On the other hand, the following 2^{nd} order damped wave was chosen by D'Alembert (see [9, p. 241]) for clearance of the variable separation method:

$$\frac{ddq}{dx^2} + \frac{\xi dq}{dx} + \frac{\zeta dq}{dt} + \lambda q + \frac{kddq}{dt^2} = 0.$$

where $dd = d^2$ and $q = q(x,t)$ is the unknown of the equation,
ξ, ζ, λ, k are functions depending on x and t.

1.2.4 Boundary Condition: Wentzell or Ventcel or Ventssel' type

The boundary conditions allow to specify the model concretely studied. Thus, scientists still explore them in various cases.

Toward the 20th century, several concepts had already been developed in functional analysis. Thus, in 1959, the work of Alexander Dmietrievich Wentzell (born in 1937) on the boundary conditions for multidimensional diffusion processes [90], led researchers to consider that kind of boundary conditions. These conditions are typical of phenomena occurring on Riemannian manifolds such as circle, sphere, and so on. The general form of Ventcel type condition satisfied by a wave u defined on manifold Γ is as follows:

$$\partial_\nu u - \Delta_\Gamma u = g,$$

where Δ_Γ is Laplacian on Γ (Laplace-Beltrami), and g a continuous function.

Since then, several researchers have taken an interest in it, notably K. Lemrabet [60–62] in the 1980s and 1990s where he studied as part of elasticity efforts, phenomena modeling the vibrations of a thin membrane, a bit sensitive at its border.

1.2.5 Energy of the system

In physical perspective, the energy of body is its capacity to perform work that is to cause a motion (in mechanic) or to affect another body's state (thermal energy, chemical energy). In this section, we give some developments of this concept in mathematical perspective.

First of all, the notion of energy in physics, dates back to a very distant era, even before the emergence of the WE (in the $18th$ century). But with the significant advance in the field of functional analysis, it has been possible to formulate the energy released by a wave problem.

The mechanical principle of the energy released by a body of mass m in motion is:

$$E_{\text{mecanic}} = E_c + E_p,$$

where E_{mecanic} is the mechanical energy (related to the general motion of the body),
E_c kinetic energy (related to the speed of the body),
E_p potential energy (related to the change in altitude of the body over a height h).

According to a specific space, the mathematical principle did the same:

- by multiplying the WE by u_t which models the wave speed;
- then by integrating over the propagation domain Ω in order to take into account changes in altitude and speed.

In 1902, Jacques Hadamard introduced in [40] the concept of the well-posedness of a Cauchy problem as a possible and determined problem that means it is physically realizable, and unique in behavior. In summary, a problem is well-posed if and only if:

- It has a solution;
- The solution is unique;
- The solution's behavior change continuously with the initial condition.

The last condition analytically indicates the existence of a constant $c > 0$ such that:

$$\|u(t)\|_H \leq c\|u(0)\|_H,$$

where $u(t)$ represents the wave at time t and H a Banach's space.

Since then, several researchers have realized that this last condition is satisfied when the energy of the system is continuously decreasing. As we can see, for instance, in these works [3, 20, 53, 55, 63, 77, 89].

However, the wave phenomenon being linked to the geometry of the propagation medium Ω, the researchers thus conducted studies on the suitable location $D \subset \overline{\Omega}$ where dissipation induces more or less strong stability.

And most of the work on exponential stability in the case where $D \subset \operatorname{int}(\Omega)$ (internal dissipation) were inspired by the methods of the cases where $D \subset \partial\Omega$ (boundary dissipation).

Case of boundary dissipation ($D \subset \partial\Omega$)

In this case, one can refer to the historical notes established in [45, Sect. 0.1], which have a significant relationship with our case. We give a little more detail on the methods used by researchers of this period.

In 1963, P. D. Lax, C. S. Morawetz and R. S. Phillips [58] showed the local exponential decay of the energy to 0 of the wave u defined on a unbounded domain $\mathbb{R}^3\backslash\overline{\Omega} = ext(\overline{\Omega})$ known as **exterior of** Ω. The obstacle Ω is star-shaped such that:

$$u_{|\operatorname{int}(\Omega)} = 0.$$

where int(Ω) denotes the interior of Ω.

The study started by the mathematical definition of incoming (*resp.* outgoing) wave of Ω as a wave which support come closer to (*resp.* go away from) Ω over time. The principle of the method used consisted of expressing the solution as a function of an orthogonal projection of incoming/outgoing waves, onto the orthogonal set of outgoing/incoming waves of Ω. These results were inspired by the work of C. H. Wilcox [95] in 1959, where the obstacle Ω was considered as a sphere and the approach was a direct analysis of the explicit solution obtained by the Euler method (separation of variables).

This was followed in 1979 by the work of G. Chen [22] on exact controllability of the damped wave over a part of boundary, early conjectured by D. L. Russell [84, p. 696]. The wave was considered on $\Omega \subset \mathbb{R}^n$ bounded and delimited by $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) - \Delta w(x,t) = 0, & \text{in } \Omega, \quad t \in (0;T),\ T > 0, \\ w(x,t) = 0, & \text{on } \Gamma_0 \\ \alpha_1 \dfrac{\partial w}{\partial n}(x,t) + \alpha_2 w(x,t) = u(x,t), & \text{on } \Gamma_1. \end{cases}$$

where $u = -\beta\frac{\partial w}{\partial t}$ is the control (boundary dissipation).

In 1981, G. Chen improved in [21] his work [22], by drawing inspiration from the works carried out by C. S. Morawetz, J. V. Ralston and W. A. Strauss, on the exterior domain [73, 86]. Chen managed to establish exponential stability by considering for a certain $x_0 \in ext(\overline{\Omega})$, the vector field $l(x) = x - x_0, x \in \overline{\Omega}$ under a certain regularity (see [52, p. 165-166]). However, in the same year, J. Lagnèse found that one of the conditions on field l was too restrictive and less natural when Ω had a smoother structure. Lagnèse discusses this in [52], where the geometric form Ω is such that there exists a vector field $l \in (C^2(\overline{\Omega}))^n$ satisfying:

$$l\,.\,\nu \leq 0 \text{ over } \Gamma_0, \quad l\,.\,\nu \geq \gamma > 0 \text{ over } \Gamma_1, \quad \text{and } \nabla l > 0 \text{ in } \overline{\Omega}.$$

These three conditions suggest the reflective (resp. absorbing) nature of Γ_0 (*resp.* Γ_1) for waves w propagating in the medium $\overline{\Omega}$.

In 1987, L. Lasiecka and R. Triggiani [57] began a search for a linear control operator F which could lead to the exponential stability of the following system:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(t,x) = \Delta w(t,x), & \text{dans } (0;\infty)\times\Omega, \\ w(t,x) = u(t,x), & \text{sur } (0;\infty)\times\Gamma \end{cases}$$

where $u = Fw_t \in L^2(0;\infty;L^2(\Gamma))$ is the control and $\Gamma = \partial\Omega \in C^1$.

Lasiecka and Triggiani carried out the study while remaining within an appropriate regularity in terms of linear control space $L^2(0;\infty;L^2(\Gamma))$. Exponential stability was established by examining an adequate variance of the released energy $E(t)$ and $E'(t)$ while taking into account the geometry of Ω as J. Lagnèse did in [52].

This section cannot be complete without mentioning the work of J. Lions, V. Komornik and E. Zuazua. These three researchers have contributed significantly to extend the concepts of controllability and stabilization. For your information, a system is exactly controllable when it is possible for a certain time $T > 0$, to bring it back to the rest state by simply acting on a specific region called the damped region.

In 1986, J. Lions [64, 66] introduced a new method for the exact controllability of reversibly evolving systems. As V. Komornik testified it in [51], Lions's new method is based on uniqueness theorems and the construction of a new Hilbert space. Thus earning the name of: **Hilbert Uniqueness Method** (HUM in short).

However, J. Lions hardly directed his research towards the methodical aspect in energy decay rate. The works of V. Komornik and E. Zuazua has contributed significantly to optimizing the results of J. Lions in that way.

We can even refer to the book [50] published by V. Komornik in 1995, which summarizes his advances inspired by, among others, J. Lions, A. Haraux, and so on.

Case of internal dissipation ($D \subset \text{int}(\Omega)$)

Whereas the boundary dissipation ($D \subset \partial\Omega$) reduces the wave amplitude at the boundary, the internal dissipation acts earlier. Thus, several researchers developed the internal case, taking into account the geometric conditions on $\overline{\Omega}$ and the methods developed in the case where $D \subset \partial\Omega$ for an efficient estimation of energy decay rate. In particular, one can cite the works [20, 47, 70–72, 76, 82, 96], which adapt them to semi-linear or even non-linear models.

In the following, we give a brief historical report about the arrival of a new approach which is one of the main stabilizing methods of this thesis.

In 1971, L. Amerio and G. Prouse [10] have made it their mission to explore the interaction between almost-periodic functions and 2^{nd} order HWE in general. A concept introduced earlier by H. Bohr [11] in 1925-1926 and then developed subsequently by several researchers (see preface, [10, p. v]).

In [10, part II], L. Amerio and G. Prouse established the strong convergence of almost-periodic solutions with any solution of the following damped wave equation in forced regime[2]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) + a_0(x)u + \beta\left(x, \frac{\partial u}{\partial t}\right) = f(x,t), & \text{in } \Omega \times (0;\infty), \\ u = 0, & \text{on } \partial\Omega \times (0;\infty) \end{cases} \tag{1.7}$$

where a_{ij}, a_0 and β are measurable functions satisfying a certain regularity properties, f designates the external force acting on u.

In 1977, M. Nakao [75], introduced an approach that analyzes the energy released over each period measured in units of time $((t, t+1), t \geq 0)$.

With this method, Nakao showed that the convergence established by L. Amerio and G. Pruse is exponential (resp. polynomial) when the dimension $dim(\Omega) = n \leq 2$ ($resp.\ dim(\Omega) > 2$).

However, a remark is immediately obvious. When the wave has a free regime[3] (f = 0), the exponential convergence is ensured independently of the dimension $dim(\Omega)$.

1.2.6 Some related works

This section contains some fairly recent work related to this thesis in terms of the models and phenomena studied, as well as the stabilization methods used.

In [78], D. C. Pereira, et al. mainly used Nakao's method to establish the exponential stability of

[2]For your information, a forced-regime wave is a wave that is affected by an external source. For example, a string swinging on a table will have its movements affected when it comes into contact with the table. It is therefore not free to move at that time. In short, these are waves that verify a nonhomgenous WE.

[3]non forced-regime

Von Kármán system [92]:

$$\begin{cases} u'' - \Delta^2 u - [u,v] + u' = f, & \text{in } \Omega \times (0,T),\ T > 0, \\ \Delta^2 v + [u,u] = 0, & \text{in } \Omega \times (0,T), \\ u = v = 0, \quad \partial_\nu u = \partial_\nu v = 0, & \text{on } \Gamma \times (0;T), \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ and $[.,.]$ represents the operator of Monge-Ampère which is defined as follows:

$$[u,v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}.$$

where $u, v \in C^2(\overline{\Omega})$.

Seeing a simple and effective way to establish the exponential stability of a coupled wave system without a source function ($f = 0$), J. A. J. Avila, et al. also used Nakao's approach in [12]. Avila et al. obtain the unconditional exponential stability of the coupled wave system with indirect control[4] and the consistency of numerical scheme of the system under a certain condition:

$$\begin{cases} u_{tt} - \Delta u + u_t + \alpha v = 0, & \text{in } \Omega \times \mathbb{R}, \\ v_{tt} - \Delta v + v_t + \alpha u = 0, & \text{in } \Omega \times \mathbb{R}, \\ u = v = 0, & \text{on } \Gamma \times \mathbb{R}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$ and $\alpha > 0$ is the coupling parameter.

The phenomena mainly considered in this thesis are related to ***feedback***[5] *phenomena,* ***transmission***[6] *phenomena, and subtle aspect of* ***delay***[7] *(not considered).*

In the case of feedback, we can refer to the work of Ammar Heminna [42, 43] in the context of elasticity and thermo-elasticity.

In [43], A. Heminna, S. Nicaise, and A. Sène established an energy decay rate of the coupled wave system (heat equation and HWE fields) with the presence of non-localized internal and

[4] see [83].

[5] Feedback occurs when there is a return in effects of the emitted wave, often due to its encounter with an obstacle or another wave. Thus, any dissipation or control mechanism of the wave causes a feedback.

[6] Wave transmission occurs when a wave changes its propagation medium. For example, when a light string attached to a heavier string is in oscillation motion, or when a membrane floating on water vibrates.

[7] This is due to the fact that the disturbing effect of the propagating wave manifests itself progressively at different times in the medium.

boundary nonlinear feedback as follows:

$$\begin{cases} u'' - div(\sigma(u)) + \alpha\nabla\theta + f(u') = 0, & \text{in } \Omega\times\mathbb{R}_+ \\ \theta' - \Delta\theta + \beta div u' = 0, & \text{in } \Omega\times\mathbb{R}_+, \\ \sigma(u) \,.\, \nu + au + g(u') = 0, & \text{on } \Gamma_2\times\mathbb{R}_+, \\ \theta_{|_\Gamma} = 0, \\ u_{|_{\Gamma_1}} = 0, \\ u(.,0) = u_0, \quad u'(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$ is open and bounded by $\Gamma \in C^2$ such that for $x_0 \in \mathbb{R}^n$ given:

$$\Gamma = \Gamma_1 \cup \Gamma_2, \quad \overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset,$$
$$\Gamma_1 = \{x \in \Gamma : (x - x_0) \,.\, \nu(x) \leq 0\}, \quad \Gamma_2 = \{x \in \Gamma : (x - x_0) \,.\, \nu(x) > 0\};$$

$u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))^\top$ represents the displacement vector field;
$\theta = \theta(x,t)$ the temperature;
$f = (f_1, \dots, f_n)^\top$ (resp. $g = (g_1, \dots, g_n)^\top$) is a vector field of functions in Ω (resp. Γ_2); σ indicates the stress tensor; $a \in C^1(\Gamma_2)$ is positive;
and $\alpha, \beta \geq 0$ the coupling parameters.

The main idea used for the exponential stability, was inspired by the work [77] of Nicaise . The method consists of imposing adequate conditions on nonlinear functions so that the stability of the nonlinear system results from the exponential stability of the associated linear system.

For those of the combined phenomena transmission-feedback, we can refer to the works [19, 47, 48]. In particular, in [47], A. Khemmoudj and Medjen studied the exponential stability of the wave system on $\Omega \subset \mathbb{R}^n, n \geq 2$ and transmitted to its boundary Γ. In addition, a linear feedback resulting from the following localized internal linear damping $a(x)u_1'$ is considered such that:

$$\begin{cases} u_1'' - \Delta u_1 + f(u_1) + a(x)u_1' = 0, & \text{dans } \Omega\times]0;+\infty[, \\ u_2'' + \partial_\nu u_1 - \Delta_T u_2 + g(u_2) + b(x)u_2' = 0, & \text{sur } \Gamma\times]0;+\infty[, \\ u_{1|_\Gamma} = u_2, \end{cases}$$

with $f, g \in C^1(\mathbb{R})$, two nonlinear functions which are globally Lipschitz or super-linear ,
$(a,b) \in L^\infty(\Omega) \times L^\infty(\Gamma)$ such that:

$$a(x) \geq a_0 > 0, \text{ over } \omega \subset \Omega, \quad b(x) \geq b_0 > 0, \text{ over } \Gamma.$$

For the analysis of asymptotic behavior, the authors adapted the multiplier method developed by J. Lions in [65] and used a unique continuation result.

In [20], M. M. Cavalcanti, I. Lasiecka, and D. Toundykov, have considered a case where the wave in $\Omega \subset \mathbb{R}^3$ is transmitted to the boundary $\Gamma = \partial\Omega \in C^2$ and two nonlinear feedback (internal and on the boundary):

$$\begin{cases} u_{tt} - \Delta u + a(x)g(u_t) = 0, & \text{in } \Omega\times]0;T[, T > 0 \\ \beta y_{tt} + b(x)g_0(y_t) + \partial_\nu u + u = \gamma\Delta_T y, & \text{over } \Gamma\times]0;T[, \\ u_{|_\Gamma} = y, \end{cases}$$

where $(a, b) \in L^\infty(\Omega) \times C^1(\Gamma)$ are non-negatives, and $\beta, \gamma > 0$,
g, g_0 represent the nonlinear feedback functions, continuous and monotone increasing such that $g(0) = g_0(0) = 0$.

Cavalcanti, Lasiecka and Toundykov showed the possibility of having uniform stability of the system under appropriate conditions between the feedback functions g, g_0 and the geometry of the domain Ω. To do this, the appropriate damped region has been constructed by combining multiple stabilization techniques linked to differential geometry.

Among the works exclusively related to delay phenomena, the following works [4, 5, 18, 35–39] can be cited in particular. Nowadays, delay terms are known for their ability to be a source of instability [24] (in feedback term). But those terms can also improve the performance of closed-loop system (see [87]) as affirmed by [45] (see section 0.3 [45]).

1.3 Motivation

This section answers the following questions:

i. **Why this thesis in particular?**

ii. **What scientific needs motivate this research?**

To begin with, it is important to understand that the main thrust of this thesis is based on the concept of stability. This does not necessarily require the formulation of a new model or even a new stabilization method. However, referring to the existing literature, some points still arouse interest. For simplicity in calculations, the works [20, 47, 63] that have dealt with, or at least speak of, the model (1.1), choose a mass density $m = 1$ on Γ_1. Which prevents us from seeing the influence of m on the energy decay rate.

Furthermore, the techniques and methods used to establish the decay rate, choose a multiplier that imposes geometric conditions on Ω and adapts the nature of the wave (absorption, reflection, expansion, ...) towards the boundaries (see [21, 22, 52]). Some even go so far as to introduce an arbitrary function as in [56] which can lead to a result of exponential stability.

However, taking all these elements into account gives rise to a multitude of parameters to consider in order to judge the contribution of m and d_0 in the exponential stability for the models considered here[8].

On the other hand, after Nakao introduced the method named after him, D. C., Pereira, et al. used it to establish an exponential decay rate for a wave satisfying the homogeneous Dirichlet condition. His goal and contribution at this time was to diversify the stabilization techniques already known. Afterwards, J. A. J. Avila [12] extends Nakao's method for a coupled wave system with two non-localized dampers, but still with homogeneous Dirichlet-type boundary conditions ($\zeta = 0$).

Thus, in a need of:

- predict the effects on the energy decay rate respect to the variation of parameters m and d_0;
- and analyze the applicability of Nakao's method for waves having a little more freedom[9] over the medium border;

this research have been initiated.

1.4 Aim and Objectives of the Research

In this section, we give the elements that we will seek to demonstrate in this thesis (aim), and the final implications of these results (objectives).

The main goal of this thesis is to establish the exponential stability of the models (1.1) and (1.2)

[8]Don't get me wrong, we are not saying that considering the geometry of Ω or the nature of the wave is unimportant; far from there, these factors are very important even in Nakao's method, which is essentially based on observability's result and the energy's continuity .

[9]The wave ζ could be nonzero over the boundary, even have almost its own dynamic.

by the Nakao method. And for this, we necessarily need to study the well-posedness of the systems (1.1) and (1.2), that is, the study of the existence, the uniqueness and the stability of the system's solution. After which, we illustrate the exponential stability (since established) for the model (1.1), in the case where $dim(\Omega) = 2$.

The objectives of this research are diverse, however, here are the main ones:

- **extension of Nakao's method**. From a mathematical and physical point of view, the homogeneous Dirichlet condition on $\partial\Omega$ is too restrictive compared to our boundary conditions. Thus, this thesis extends Nakao's method, for cases where a little more freedom is offered to the wave on a part of the boundary $\Gamma_1 \subset \partial\Omega$ completely disjoint from the rest of the boundary of Ω.

- **popularization of theoretical results for experimental physicists**. The great thing about Nakao's method is that it provides a fixed decay rate. This makes it easier to determine the effect of varying the parameters involved.

1.5 Methodology of Research

In this section clarifies the processes to achieve the goal of this thesis. As seen previously, the aim of this thesis is more methodical than anything else. So, before applying Nakao's method for exponential stability, we first used a combination of **frequency domain** method of Prüss [81], the construction of cut-off function of J. Lions [65], the multiplier technique of V. Komornik [49], and finally a result of K. Liu [67].

We would like to point out that this combination was inspired by the work of M. Akil and Z. Hajjej [3]. Here are the methods used in this thesis:

- **Method of semi-groups:** It begins by reformulating the system studied as a first-order Cauchy problem depending on an operator A defined on a well-defined Hilbert space. Then, an analysis of the properties of the system operator on its domain of definition $D(A)$, is done to ensure its bijectivity on D(A) which guarantees the existence and uniqueness of the solution.

- **Method of Lions-Komornik-Prüss-Liu:**
  This method allows us to establish exponential stability by analyzing the properties of

the operator A resolvent. To begin with, thanks to the compactness of the operator A, we demonstrate the strong stability of the system, i.e. that the energy $E(t) \to 0$, when $t \to +\infty$. To do this, we show that the spectrum of A does not contain any pure imaginary thanks to a proof by contradiction, resulting from a unique continuation principle.

Then, we impose the geometric condition of the Piecewise Multiplier Geometric Condition (PMGC in short) on the damped region ω. This consists of dividing the damped region into several pieces on which we define cut-off functions introduced by J. Lions [65] allowing us to locally estimate the energy terms. Finally, by a proof by contradiction, we establish that the sup of A resolvent inverse for pure imaginaries tends towards 0, which according to J. Pruss [81] and K. Liu [67], indicates an exponential stability.

- **Method of Nakao :**

  This method is based on the energy analysis of dissipative systems. It consists of studying the energy released by the system on $\overline{\Omega}$ per unit of time. So after having established the decay of the system's energy by the already known technique (see [3, 89]), the study seeks to show that the maximum energy released on duration T can be estimated by the energy lost on T by the system. To achieve the latter, the method combines the multiplier technique and integral estimations leading to the appearance of the maximum energy and the energy lost on T (see [12, 75, 78]).

1.6 Thesis Outline

In this section we give the general structure of this thesis. The thesis includes a general introduction, four chapters, a general conclusion and perspectives.

- **Chapter 1 :**

  This chapter is devoted to the general introduction of the thesis where we discuss the different concepts (modeled phenomena, methods used, ...) and works related to this thesis.

- **Chapter 2 :**

  In this chapter, we recall some useful notions to achieve the goal of this thesis. In particular, on functional analysis, Sobolev spaces, Lebesgue spaces L^p, hyperbolic PDEs of order 2, the theory of semigroups, some concepts of stability, the method of finite differences.

- **Chapter 3 :**

  This chapter studies the exponential stability of the system (1.1) of the first model (1.1) by the Lions-Komornik-Pruss-Liu method. Firstly, we establish the existence and uniqueness of the solution by the semigroup method, then we demonstrate that the energy of the system is dissipative by the method of energy. Then, a strong stability result is demonstrated thanks to the Arendt-Batty theorem and a Carleman estimation. Finally, the exponential stability of the system is established while the damped region $\omega \subset \Omega$ is under the PMGC condition.

- **Chapter 4 :**
  This chapter deals with the exponential stability of the system (1.2) of second model (1.2) by the Nakao method. To do this, we study the well-posedness of the system (1.2) in the same way as in **Chapter 3**. Then, using Nakao's method, an exponential decay rate of energy is obtained. This exponential decay is then illustrated in the two-dimensional case using the finite difference method.

- **Chapter 5 :**
  This final chapter analyzes the asymptotic behavior of the system (1.1) using Nakao's method. We therefore use the same procedures as in **Chapter 4**. The difference between this chapter and the two previous ones lies in the physical interpretation part that we will add.

In summary, the general conclusion of the thesis is given; followed by a list of some difficulties encountered that could be the subject of future research. Finally, the document ends with the bibliography.

1.7 List of Publications & Conferences

This thesis is presented as a series of one submitted manuscript and two published manuscripts (see [13, 14]) and two international conferences for the research's communication.

- *Barry., K. L., & Laoubi., K.,*: Stabilization and Observability Analysis of the Wave Equation Subject to Internal and Dynamic Conditions, submitted on 2023.

- *Barry., K. L., & Laoubi., K.,* Stabilization of the wave equation with internal damping and numerical simulations, International Journal of Control, DOI: 10.1080/00207179.2025.2461595, (2025).

- *Barry., K. L., & Laoubi., K.,*: Theoretical and numerical stabilization of the wave equation subject to internal and dynamic controllers. Arab. J. Math., https://doi.org/10.1007/s40065-025-00536-w, (2025).

- *Barry., K. L.*: Longtime Behavior of Wave Equation with Kinetic Boundary Condition, 4^{th} Baskent International Conference On Multidisciplinary Studies, ISBN:978-625-367-236-2, August 04-06, 2023, URL: www.izdas.org

- *Barry., K. L.:* Exponential Stability of Locally Damped Wave and Illustration, International Conference on Mathematics and Computers with Applications, July 16-18, 2025

2

PRELIMINARIES

This chapter serves as a reminder of several required concepts to gain a thorough comprehension of the analytical research carried out in this thesis. These reminders are arranged as follows: functional analysis, Lebesgue spaces L^p, Sobolev spaces, second order hyperbolic PDEs, some concepts of stability, and finite-differences.

For more information and proof we direct readers to the references [1, 15, 23, 28, 30, 34, 45, 67, 81, 93, 94, 98], and those cited therein.

2.1 Functional Analysis

Here, we present certain basic notions of functional analysis that will be helpful for the rest of this chapter.

2.1.1 Brief Introduction

In a general way, examining an X-space's topology leads back to examinate its structure. This entails to consider its portions currently known as **open set** in X, such that:

a. Both the empty set $\varnothing$ and X are open sets in X,

b. Any finite intersection of open sets in X is an open set in X,

c. And any union of open sets in X is still an open set in X.

In summary, the basical characteristics that make up connected a geometrical object. Therefore, a single object (space) may have several structures (topology) depending on the open sets considered.

Now, let's consider two different collections of open set in X, denoted by, τ_1 and τ_2 such that:

$$[\tau_1 \subset \tau_2] \Longleftrightarrow [\forall \mathcal{O} \in \tau_1 \ : \ \mathcal{O} \in \tau_2].$$

Then the topology or structure τ_1 (*resp.* τ_2) is **coarser** (*resp.* **finer**) than τ_2 (*resp.* τ_1). In addition, τ_1 (*resp.* τ_2) is also said to be **weaker** (*resp.* **stronger**) than τ_2 (*resp.* τ_1).

The topological properties that we found most important in this reminder are either that the open spaces are indexed by continuous forms $f : X \to \mathbb{R}$ (dual of X) where X is a Banach space.

2.1.2 Dual Space

Let $(X, \|.\|_X)$ be a Banach space. We call **dual of X**, and denotes by X^*, all the continuous and linear forms on X that is

$$X^* = \{f \ : \ X \longrightarrow \mathbb{R}, \text{ continuous and linear}\}.$$

The inner norm of X^* is defined as:

$$\|f\|_{X^*} = \underset{x\in X, \|x\|\leq 1}{sup} |\langle f, x\rangle|,$$

where $\langle f, x\rangle = f(x)$ represents the scalar product between $f \in X^*$ and $x \in X$.

2.1.3 Strong, weak and weak-* convergences

Definition 2.1.1. *(Strong Topology)*
Let $(X, \|.\|_X)$ be a Banach space. Then X has strong topology where its elements are open balls associated to the norm $\|.\|_X$.

Definition 2.1.2. *(Strong convergence)*
Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. We say that $(x_n)_{n\in\mathbb{N}}$ ***converge strongly*** *in X, if there exists $x \in X$ such that:*

$$\|x_n - x\|_X \to 0, \quad when\ n \to +\infty.$$

Then we write,

$$x_n \to x \in X.$$

Definition 2.1.3. *Weak topology $\sigma(X, X^*)$*
Let $(X, \|.\|_X)$ be a Banach space. We set a collection of continuous linear maps $\left(\varphi_f\right)_{f\in X^*}$ such that:

$$\begin{aligned} \varphi_f : X &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi_f(x) = \langle f, x\rangle \end{aligned}.$$

The weak topology $\sigma(X, X^*)$ on X, is the coarsest topology on X associated to $\left(\varphi_f\right)_{f\in X^*}$.

Definition 2.1.4. *(Weak convergence)*
*Let $(x_n)_{n\in\mathbb{N}}$ be sequence of X elements. We say that $(x_n)_{n\in\mathbb{N}}$ **converge weakly** in X, if there exists $x \in \sigma(X, X^*)$ such that:*

$$\forall f \in X^*,\ \varphi_f(x_n) \to \varphi_f(x), \quad when\ n \to +\infty.$$

Then we write,

$$x_n \rightharpoonup x \in \sigma(X, X^*).$$

Now, here are some usual properties in the weak topology $\sigma(X, X^*)$:

Proposition 2.1.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X elements. Then, we have:*

i. $[x_n \rightharpoonup x \in \sigma(X, X^*)] \iff [\langle f, x_n\rangle \to \langle f, x\rangle, \quad \forall f \in X^*]$

ii. $[x_n \to x \in X] \implies [x_n \rightharpoonup x \in \sigma(X, X^*)]$

iii. $[x_n \rightharpoonup x \in \sigma(X, X^*)] \implies [\|x_n\|_X < +\infty, \quad \|x\|_X \leq lim\ inf\|x_n\|_X]$

iv. $[x_n \rightharpoonup x \in \sigma(X, X^*), \quad f_n \to f \in X^*] \implies [\langle f_n, x_n\rangle \to \langle f, x\rangle].$

Proof. See Brezis [15, p. 58]. □

Using the same technique of construction as for $\sigma(X, X^*)$, a more coarsest topology on dual spaces is defined.

Definition 2.1.6. *(Weak-* topology)*
Consider the collection of linear functional $(\varphi_x)_{x\in X}$ such that:

$$\begin{aligned} \varphi_x : X^* &\longrightarrow \mathbb{R} \\ f &\longmapsto \varphi_x(f) = \langle f, x\rangle \end{aligned}.$$

The weak- (or weak star) topology $\sigma(X^*, X)$ on X^*, is the coarsest topology on X^* associated to $(\varphi_x)_{x\in X}$.*

Remark 2.1.7. *The star ($*$) is here to remind that this topology is defined only on dual spaces.*

Definition 2.1.8. *(Weak-* convergence)*
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of X^* elements. We say that f_n ***converge weakly*** *in the weak-* topology $\sigma(X^*, X)$, if there exists $f \in \sigma(X^*, X)$ such that:*

$$\forall x \in X,\ \varphi_x(f_n) \to \varphi_x(f), \quad lorsque\ n \to +\infty.$$

Then we write,

$$f_n \overset{*}{\rightharpoonup} f \in \sigma(X^*, X).$$

Remark 2.1.9. *The concept of weak-* topology is not unique when we refer to the remark(2.1.7). For instance, $\sigma(X^*, X^{**})$ is also a weak-* topology on X^*.*

Therefore, $\sigma(X^*, X^{**})$ is finer than $\sigma(X^*, X)$ since $X \subset X^{**}$. The two weak- topology coincide when X is reflexive, and that is $X^{**} = X$.*

Proposition 2.1.10. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of X^* elements. Then, we have:*

i. $\left[f_n \overset{*}{\rightharpoonup} f \in \sigma(X^*, X)\right] \iff [\langle f_n, x\rangle \to \langle f, x\rangle, \quad \forall x \in X]$

ii. $[f_n \to f \in X^*] \Longrightarrow [f_n \rightharpoonup f \in \sigma(X^*, X^{**})]$

$$[f_n \rightharpoonup f \in \sigma(X^*, X^{**})] \Longrightarrow [f_n \rightharpoonup f \in \sigma(X^*, X)]$$

iii. $\left[f_n \overset{*}{\rightharpoonup} f \in \sigma(X^*, X^{**})\right] \Longrightarrow [\|f_n\|_{X^*} < +\infty, \quad \|f\|_{X^*} \le lim\ inf \|f_n\|_{X^*}]$

iv. $[f_n \rightharpoonup f \in \sigma(X^*, X), \quad x_n \to x \in X] \Longrightarrow [\langle f_n, x_n\rangle \to \langle f, x\rangle].$

Proof. See Brezis [15, p. 63]. □

2.1.4 Theorem of Banach-Alaoglu

Here, we present a result of compactness from [30, p. 5] (see also [15, p. 66]) that will be required later, in chapters 4 and 5.

Definition 2.1.11. *Let $K \subset X$ a compact subset of Banach space X. We say that K is* ***weakly sequentially compact*** *if for any sequence $(x_n)_n \subset K$, there is a convergent subsequence $(x_{n_k})_{n_k}$ such that:*

$$x_{n_k} \rightharpoonup x \in K.$$

Theorem 2.1.12. *(Reflexive case)*
Let X be a reflexive Banach space (that is, $X^{**} = X$). Then the unit closed ball B of X, defined as:

$$B = \{x \in X \,:\, \|x\|_X \leq 1\},$$

is weakly sequentially compact.

2.2 L^p SPACES

Since we are conducting research on wave modeled as differential equation (PDE), it follows that the solution analysis will depend on that of a certain class of integral. As a result, this section briefly recalls certain ideas about lebesgue spaces L^p, which generalized the concept of integrability for functions raised to the p-th power.

Definition 2.2.1. *Let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}^*$ be measurable, and $p \in [1;\infty]$. The Lebesgue spaces $L^p(\Omega)$ are vector space of class of measurable functions $f : \Omega \to \mathbb{C}$, such that:*

$$\|f\|_{L^p(\Omega)} < +\infty,$$

where $\|.\|_{L^p(\Omega)}$ denotes the inner norm of $L^p(\Omega)$, defined as:

$$\|f\|_{L^p(\Omega)} = \begin{cases} \int_\Omega |f(x)|^p \mathrm{d}x < +\infty, & \text{if } 1 \leq p < +\infty \\ \sup_{x\in\Omega} ess|f(x)| = inf\{M \geq 0 \,:\, |f(x)| \leq M, \text{ a.e. in } \Omega\} < +\infty, & \text{if } p = +\infty, \end{cases}$$

Notation 1. *Let $p \in [1;\infty]$. Then $q \in [1;\infty]$ is called the Hölder conjugate or simply conjugate of p if:*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

By convention, $\frac{1}{\infty} = 0$. Then ∞ is the conjugate of 1, and conversely, 1 is the conjugate of ∞.

For more readability, let's write $\|.\|_p$ instead of $\|.\|_{L^p(\Omega)}$.

2.2.1 SOME PROPERTIES

Firstly, $L^p(\Omega)$ is a complete space for its inner norm $\|.\|_p$, and that is, $(L^p(\Omega), \|.\|_p)$ is a Banach space. In particular, $L^2(\Omega)$ is of Hilbert for the inner scalar product:

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\mathrm{d}x, \quad \forall f, g \in L^2(\Omega).$$

Lemma 2.2.2. *Hölder's inequality Let $p \in [1;\infty]$ and q be its conjugate. Then, for all $(f,g) \in L^p(\Omega) \times L^q(\Omega)$, we have:*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_q. \tag{2.1}$$

REFLEXIVITY, SEPARABILITY, AND DUAL OF $L^p(\Omega)$

Let $p \in [1;\infty]$. Assume that $\Omega \subset \mathbb{R}^n,\ n \in \mathbb{N}^*$ is measurable. Then, the space $L^p(\Omega)$ is:

i. reflexive and separable if $p \in]1;+\infty[$. Moreover, we have:

$$(L^p(\Omega))^* = L^q(\Omega)$$

where q is the conjugate of p.

ii. non-reflexive and separable if $p = 1$. Moreover, we have:

$$(L^1(\Omega))^* = L^\infty(\Omega)$$

iii. non-reflexive, non-separable if $p = \infty$. However, the dual space of $L^\infty(\Omega)$ only satisfies the following inclusion (see [15, p. 103]):

$$L^1(\Omega) \subset (L^\infty(\Omega))^*.$$

Theorem 2.2.3. *(Riesz representation theorem)*
Let $p \in]1;\infty[$ and q be its conjuguate. For $\varphi \in (L^p(\Omega))^*$, there exists a unique $u \in L^q(\Omega)$ such that:

$$\varphi(f) = \int_\Omega u(x)f(x)\mathrm{d}x, \quad \forall f \in L^p(\Omega).$$

Moreover,

$$\|u\|_{L^q(\Omega)} = \|\varphi\|_{(L^p(\Omega))^*}.$$

Remark 2.2.4. *This result is very important, in the absence of which it would have been impossible to have an identity between the dual space $(L^p(\Omega))^*$ and the functionnal space $L^q(\Omega)$.*

2.3 SOBOLEV SPACES

For many years, researchers understood that $C^k, k \geq 1$ regularity, was too strong to be satisfied for many solution of PDEs. So, that was not appropriate for the study of such a PDE. Therefore, the implementation of a new type of less strong spaces in regularity has been introduced, in the sense of distributions known as **weak derivative spaces**. The spaces considered, are named after the mathematician Sergueï Lvovitch Sobolev.

Sobolev Spaces $W^{m,p}$

Notation 2. *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We set the α-th derivative operator:*

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \dots \partial {x_n}^{\alpha_n}},$$

where $|\alpha| = \sum_{i=1}^n |\alpha_i|$.

Definition 2.3.1. *(weak derivative)*
*Let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}^*$. Let $u, v \in L^1_{loc}(\Omega)$. We say that $v = D^\alpha u$ is the $|\alpha| - th$ **weak derivative** of u if:*

$$\int_\Omega u(x) D^\alpha \varphi(x) \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x)\mathrm{d}x, \quad \forall \varphi \in \mathscr{D}(\Omega),$$

with $\mathscr{D}(\Omega)$ as the test functions space.

Definition 2.3.2. *(Spaces $W^{m,p}(\Omega)$)*
Let $m \geq 1$ and $p \in [1; \infty]$. We called Sobolev space $W^{m,p}(\Omega)$, any element of $L^p(\Omega)$ having all its weak derivatives in $L^p(\Omega)$ up to the order m, that is:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \; : \; D^\alpha u \in L^p(\Omega), \; \forall |\alpha| \leq m\},$$

The inner norm defined on $W^{m,p}(\Omega)$, is:

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum\limits_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, & si \; p \in [1; \infty[\\ \max\limits_{|\alpha| \leq m} \|D^\alpha\|_{L^\infty(\Omega)}, & si \; p = \infty. \end{cases}$$

Notation 3. *When $p = 2$, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$.*

Remark 2.3.3. *According to its definition, the space $W^{m,p}$ satisfies:*

$$W^{m,p} \subset L^p$$

. Consequently, $W^{m,p}$ and L^p have some similar properties.

Proposition 2.3.4. *(Reflexivity and Separability) Let $m \geq 1$ and $p \in [1; \infty]$. The following properties hold:*

i. $(W^{m,p}, \|.\|_{W^{m,p}})$ is of Banach.

In particular, H^m is a separable Hilbert space for its inner scalar product:

$$\langle u, \; v\rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) \mathrm{d}x, \quad \forall u, v \in H^m(\Omega).$$

ii. $W^{m,p}$ *is reflexive, if* $p \in]1;\infty[$.

iii. $W^{m,p}$ *is separable, if* $p \in [1;\infty[$.

iv. $W^{m,\infty}$ *is non-reflexive, and non-separable.*

Definition 2.3.5. *(Continuous Embedding)*
Let E, F *be two normed vector spaces. We say that* E *is* ***continuously embedded*** *in* F*, if there exist an inclusion map* $i : E \to F$ *and a constant* $C > 0$*, such that:*

$$\|i(x)\|_F \leq C\|x\|_E, \quad \forall x \in E.$$

Then, we write $E \hookrightarrow F$.

Corollary 2.3.6. *Let* E, F *be two normed vector spaces.*
The following propositions are equivalent:

i. $E \hookrightarrow F$.

ii. If the sequence $(x_n)_n \subset E$ *is strongly convergent in* E*, then the sequence* $(i(x_n))_n$ *converge strongly in* F.

Definition 2.3.7. *(Compact Embedding)*
Let E, F *be two normed vector spaces. We say that* E *is* ***compactly embedded*** *in* F*, if there is a compact inclusion map* i *from* E *into* F*, that is:*

i. $E \hookrightarrow F$,

ii. For any bounded set B *in* E, $\overline{i(B)}$ *is compact.*

Then, we write $E \hookrightarrow\hookrightarrow F$.

Corollary 2.3.8. *Let* E, F *be two normed vector spaces.*
The following propositions are equivalent:

i. $E \hookrightarrow\hookrightarrow F$

ii. For all bounded sequence $(x_n)_n \subset E$*, there exists a subsequence* $(x_{n_k})_{n_k} \subset (x_n)_n$ *which converge strongly in* F.

Sobolev Spaces $W_0^{m,p}$

After introducing the concept of spaces $W^{m,p}(\Omega)$, an analysis of this space via the elements of $\mathscr{D}(\Omega)$ was not conclusive due to a lack of density result. In particular, for many open subsets of $\mathbb{R}^n$ where most of the physical phenomena happen. Thus, the mathematicians introduced another class of Sobolev spaces.

Definition 2.3.9. *Let $m \geq 1$ and $p \in [1;\infty[$.*
The space $W_0^{m,p}(\Omega)$ (resp. H_0^m) is the closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$ (resp. $H^m(\Omega)$).

That is mathematically expressed as:

$$W_0^{m,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{W^{m,p}(\Omega)}, \quad H_0^m(\Omega) = \overline{\mathscr{D}(\Omega)}^{H^m(\Omega)}$$

Corollary 2.3.10. *Let $m \geq 1$ and $p \in [1;\infty[$. The following propositions hold:*

i. $W_0^{m,p}(\Omega)$ is a separable Banach space for the topology, thus induced by the norm $\|.\|_{W^{m,p}(\Omega)}$.

ii. $W_0^{m,p}(\Omega)$ is reflexive, if $p \in]1;\infty[$.

In particular, $H_0^m(\Omega)$ is reflexive and separable Banach space for the induced topology by $\|.\|_{H^m(\Omega)}$.

Proposition 2.3.11. *Assume that $\Omega \subset \mathbb{R}^n$ such that $\Gamma = \partial\Omega \in C^1$. Let*

$$u \in W^{1,p}(\Omega) \cap C^1(\overline{\Omega}), \quad \textit{with } p \in [1;\infty[.$$

Then, we have the following equivalence:

$$u \in W_0^{1,p}(\Omega) \Longleftrightarrow u_{|\Gamma} = 0.$$

Theorem 2.3.12. *(Trace)*
The trace operator $T : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$, $\Gamma = \partial\Omega$, such that:

$$Tu = u_{|_\Gamma}, \quad \forall u \in H^1(\Omega),$$

is linear, continuous and surjective.

Lemma 2.3.13. *(H_0^1 Poincaré's Inequality)*
Assume that Ω is an open bounded in one direction. Then there exists an optimal constant $c_p = c_p(\Omega) > 0$ depending on Ω such that:

$$\|u\|_{L^2(\Omega)} \leq c_p \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{2.2}$$

*where c_p is called **poincare's constant**.*

Proof. See [1, 15]. □

Corollary 2.3.14. *(Another version)*
Assume that the conditions of lemma (2.3.13) still hold.
We set:

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) \;:\; u_{|\Gamma_0} = 0\right\},$$

where $\Gamma_0 \subset \Gamma = \partial\Omega$, *is relatively open such that* $\text{meas}(\Gamma_0) \neq 0$.
Then, the poincare's inequality given in (2.2) *still hold for all* $u \in H^1_{\Gamma_0}(\Omega)$.

Proof. This is obvious from the proof of lemma (2.3.13). □

2.4 Second order hyperbolic PDE

In this section, we recall some details about second-order hyperbolic PDEs so that our readers become more familiar with the concept.

2.4.1 Definition

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. We say that $u = u(x), x \in \Omega$ satisfies the second-order PDE, any relation F in u such that:

$$F(x, u, \partial^1 u, \partial^2 u) = 0 \tag{2.3}$$

where $\partial^{|\alpha|} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ and $|\alpha| = \sum_{i=0}^{n} \alpha_i$, for all multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Thus, the PDE is said to be linear (resp. non-linear) when the relation F is linear (resp. non-linear).

2.4.2 Classification and Types of linear 2^{nd} order PDE

Let $(x, y) \in \Omega_1 \times \Omega_2 = \Omega$. In the linear case, the PDE (2.3) can be written as follows:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{2.4}$$

with $a, b, c, d, e, f, g \in \mathbb{R}$ such that a, b, c nonzero at the same time.

The type of the PDE (2.4) depends on the sign of the discriminant $\Delta = b^2 - ac$:

- Elliptic EDP $\Delta < 0$: This type is generally used for stationary phenomena, that is, phenomena that are constant over time.

- Parabolic EDP $\Delta = 0$: This case models the transient evolution of irreversible phenomena associated with diffusion processes.

- Hyperbolic EDP $\Delta > 0$: This case, which mainly concerns our thesis, models transport or propagation phenomena.

2.4.3 BOUNDARY CONDITIONS

Physical phenomena occur in most cases over bounded domains. One or more boundary conditions are therefore necessary to maintain the phenomenon in a certain equilibrium and thus specify exactly the phenomenon to be modeled.

There are a multitude of boundary conditions, most of which model the interaction of physical forces or geometric constraints on the boundary. For example, a Dirichlet boundary condition (ζ = constant on the boundary) is a geometric constraint forcing boundary constancy.

Boundary conditions			
Type	Static/Dynamic	Modeling Effect	Condition
Dirichlet	can be both	rigid border and/or fixed	$\zeta = g,\, g \in C^0(\partial\Omega)$
Neumann	can be both	slightly rigid border	$\partial_\nu \zeta = g,\, g \in C^0(\partial\Omega)$
Robin ou Fourier	can be both	less rigid border	$\alpha\zeta + \beta\partial_\nu\zeta = g,\ \alpha, \beta, g \in C^0(\partial\Omega)$ and $\alpha, \beta \neq 0$
Wentzell	can be both	flexible border and neglected inertia	$\alpha\partial_\nu\zeta - \beta\Delta_T\zeta = g,\, \alpha, \beta \neq 0,$ $g \in C^0(\partial\Omega)$
Wentzell dynamic	Dynamic	flexible border with inertia	$a\zeta_{tt} + b\partial_\nu\zeta - c\Delta_T\zeta = g,$ $a, b, c \neq 0,\ g \in C^0(\partial\Omega)$

2.5 THEORY OF SEMIGROUPS

Definition 2.5.1. *(Semigroup)*
Let E be a Banach space. We call ***semigroup*** *of bounded linear operators on E, a one-parameter family* $(G(t))_{t\in\mathbb{R}_+}$ *of* $\mathscr{L}(E)$ *elements, such that:*

1. $G(0) = I$

2. $G(t+s) = G(t).G(s)\ \forall t, s \geq 0$

Moreover, such a semigroup is said to be **uniformly continu** or **strongly continu** if:

$$\lim_{t \to 0} \|G(t) - I\| = 0$$

And then, $G(t)$ is called $\mathscr{C}_0$ semigroup [1].

Definition 2.5.2. *(Inifinitesimal Generator)*
Let $S(t)$ be $\mathscr{C}_0$ semigroup on H. The inifinitesimal generator A of $S(t)$ is defined by:

$$Ax = \lim_{t \longrightarrow 0} \frac{1}{t}[S(t)x - x]$$

whenever the limit exists. The domain of A, denoted by $D(A)$, is defined by:

$$D(A) = \left\{x \in H,\ \lim_{t \longrightarrow 0} \frac{1}{t}[S(t)x - x] \text{ exists}\right\}$$

Theorem 2.5.3. *(Hille-Yosida)*
Let E be a Banach space. Assume that A is a linear operator (not necessarily bounded) in E. Then A generates a $\mathscr{C}_0$ semigroup of contraction on E if and only if:

1. *A is closed and $D(A)$ is dense in E.*

2. *The resolvent set $\rho(A) = \{\lambda \in \mathbb{C}, \lambda I - A$ has an inverse from $D(A) \to E\}$ contains $\mathbb{R}_+^*$, Moreover $\forall \lambda > 0$, we have*

$$\|R(\lambda, A)\| = \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$$

Theorem 2.5.4. *(Lumer-Philips)*
Let A be a linear operator with dense domain $D(A)$ in E. So,

1. *If A is dissipative and there exists $\lambda_0 > 0$ such that $Im(\lambda_0 I - A) = E$, then A generates a $\mathscr{C}_0$ semigroup of contraction.*

2. *Conversely, if A generates a $\mathscr{C}_0$ semigroup of contraction, then $Im(\lambda_0 I - A) = E$, with $\lambda_0 > 0$ and A is dissipative.*

In addition,

$$Re < x^*, Ax > \leq 0, \quad \forall (x, x^*) \in D(A) \times F(x),$$

where $F(x)$ is the dual elements associated to x, defined by:

$$F(x) = \{x^* \in E^*, \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}.$$

2.6 Concepts of Stability

Definition 2.6.1. *The semigroup $T(t) = e^{At}$ is said to be exponentially stable if there exist two constants $\alpha > 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

Definition 2.6.2. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ is said to be:*

1. *Uniformly exponentially stable if there exists $\varepsilon > 0$ such that:*

$$\lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0.$$

2. *Uniformly stable if:*

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

3. *Strongly stable if:*

$$\lim_{t\to\infty} \|T(t)x\| = 0, \quad \forall x \in E.$$

4. *Weakly stable if:*

$$\lim_{t\to\infty} \langle T(t)x, x^* \rangle = 0, \quad \forall\, x \in E \quad and \quad x^* \in E^*.$$

Proposition 2.6.3. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$, the following assertions are equivalent:*

1. *$(T(t))_{t\geq 0}$ is uniformly exponentially stable.*

2. *$(T(t))_{t\geq 0}$ is uniformly stable.*

3. *There exists $\varepsilon > 0$ such that $\lim_{t\to\infty} e^{\varepsilon t}\|T(t)x\| = 0$, $\forall x \in X$.*

Proof. It is clear that (1) implies (2) and (3).
$e^{\omega_0 t} = r(T(t)) \leq \|T(t)\|$, $\forall t \geq 0$ because:

$$\omega_0 = \inf_{t\geq 0} \frac{1}{t} \log \|T(t)\|.$$

Then

$$\omega_0 \leq \frac{1}{t} \log \|T(t)\|, \forall t \geq 0$$

Thus, $e^{\omega_0 t} \leq \|T(t)\|, \quad \forall t \geq 0$:

(2) $\Longrightarrow \omega_0 < 0, \big(\omega_0 := \inf\{\omega \in \mathbb{R}, \quad \lim_{t\to 0} e^{-\omega t}\|T(t)\| = 0\}\big)$, therefore:

$$\exists\, \omega \,:\, \omega_0 < \omega < 0 \,:\, \lim_{t\to 0} e^{-\omega t}\|T(t)\| = 0$$

Choose $\varepsilon = -\omega$

Then: $\lim_{t\to 0} e^{\varepsilon t}\|T(t)\| = 0$ thus (1).

If (3) is satisfied, then $\big(e^{\varepsilon t}T(t)\big)_{t\geq 0}$ is strongly hence uniformly bounded,

then $\exists\, \alpha > 0$ such that:

$$\begin{aligned} \left\|e^{\varepsilon t}T(t)\right\| \leq \alpha \Longrightarrow \quad & e^{\varepsilon t}\|T(t)\| \leq \alpha \\ \Longrightarrow \quad & e^{\frac{E}{2}}\|T(t)\| \leq \alpha e^{-\frac{\varepsilon}{2}t} \end{aligned}$$

which implies $\lim_{t\to\infty} e^{\frac{\varepsilon}{2}t}\|T(t)\| = 0$ thus (1). □

Proposition 2.6.4. *Let $(T(t))_{t\geq 0}$ be a semigroup with generator A, then $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if*

$$\omega_0 < 0.$$

Proposition 2.6.5. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$, the following assertions are equivalent:*

1. *$\omega_0 < 0$, it means, $(T(t))_{t\geq 0}$ is uniformly exponentially stable.*
2. *$\lim_{t\to\infty} \|T(t)\| = 0$.*
3. *$\|T(t_0)\| < 1$ for some $t_0 > 0$.*
4. *$r(T(t_1)) < 1$ for some $t_1 > 0$.*

(See [28])

Theorem 2.6.6. *Let $T(t) = e^{\mathscr{A}t}$ be a C_0-semigroup on a Hilbert space. Then $T(t)$ is exponentially stable if and only if:*

$$\sup\{\operatorname{Re}\lambda; \lambda \in \sigma(\mathscr{A})\} < 0 \tag{2.5}$$

and

$$\sup_{\operatorname{Re}\lambda\geq 0} \left\|(\lambda I - \mathscr{A})^{-1}\right\| < \infty. \tag{2.6}$$

Theorem 2.6.7. *Let $T(t) = e^{\mathscr{A}t}$ be a C_0 contraction semigroup on a Hilbert space. Then $T(t)$ is exponentially stable if and only if:*

$$\rho(\mathscr{A}) \supseteq \{i\beta,\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{2.7}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \left\|(i\beta I - \mathscr{A})^{-1}\right\| < \infty. \tag{2.8}$$

Proof. See [69]. □

Theorem 2.6.8. ***(Gearhart 1978, Prüss 1984, Greiner 1985)***
Let $\tau = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space H with infinitesimal generator $\mathscr{A}$, then $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is contained in the resolvent set $\rho(\mathscr{A})$ of the generator $\mathscr{A}$ with the resolvent satisfying

$$M := \sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, \mathscr{A})\| < \infty. \tag{2.9}$$

Proof. See [27]. □

Now, let's give another way to show the exponential stability.

Proposition 2.6.9. *(From Nakao [75, p. 339-340])*
Let $\phi(t)$ be a bounded positive function on $\mathbb{R}_+$ satisfying for some constant $k \in]0, 1[$:

$$\phi(t+1) \leq (1-k)\phi(t), \quad \forall t \geq 0. \tag{2.10}$$

Then, we have:

$$\phi(t) \leq Me^{-k't},$$

where $k' = -log(1-k)$ and $M = \frac{1}{1-k} \underset{t\in[0;1]}{max}\, \phi(t)$.

Proof. For $t > 1$, there exists an integer n such that $n \leq t \leq n+1$. Then using (2.10), the following inequality chain follows:

$$\phi(t) \leq (1-k)\phi(t-1) \leq (1-k)^2\phi(t-2) \leq \ldots \leq (1-k)^n\phi(t-n).$$

Consequently,

$$\phi(t) \leq (1-k)^n\phi(t-n), \quad \forall t \in [n; n+1].$$

Knowing that $t \in [n; n+1]$, then $t - n \in [0; 1]$ and it follows that:

$$\phi(t) \leq (1-k)^n M_0 = M_0 e^{-nk'}, \tag{2.11}$$

where $k' = -log(1-k) > 0$ and $M_0 = \max_{t\in[0;1]} \phi(t)$.
Using the fact that $n \mapsto e^{-nk'}$ decreases with n and $n - 1 \leq t - 1 \leq n$, it follows for all $t > 1$:

$$\begin{aligned} e^{-nk'} &\leq e^{-k'(t-1)} \\ &\leq e^{-k't} e^{k'} = \frac{1}{1-k} e^{-k't}. \end{aligned}$$

Using this final result in (2.11), we obtain:

$$\phi(t) \leq \frac{M_0}{1-k} e^{-k't} = M e^{-k't},$$

where $M = \frac{M_0}{1-k}$.
The proof is complete. □

2.7 Finite differences method

In this section, we focus solely on the "finite differences method". This technique replaces the derivatives in the PDE with divided differences, based on the function's values at discrete points forming a mesh (see [23, 93, 98]. The main advantage of this method lies in its simplicity of notation and ease of access for numerical computations. It primarily relies on two aspects:

1. Discretization of derivation or differentiation operators: Which transforms the differential equation into a set of algebraic equations.

2. Convergence of the numerical scheme: Essential to ensure that the numerical solution approaches the exact solution as the mesh is refined.

The main drawback of this method is its limitation to relatively simple geometries and the difficulty in handling certain boundary conditions, particularly Neumann conditions, where managing normal derivatives can be problematic.

2.7.1 Domain Discretization

Let $u : \mathbb{R} \to \mathbb{R}$ be a smooth function defined on an interval $[a, b]$. We introduce a uniform mesh with step size $h > 0$ defined by:

$$x_i = a + ih, \quad i = 0, 1, \ldots, N, \quad \text{where } h = \frac{b-a}{N}.$$

The function u is evaluated only at the mesh points x_i, and its derivatives are approximated using finite difference operators.

2.7.2 Finite Difference Approximations

First-Order Derivative

Three common finite difference approximations for the first derivative $u'(x)$ are:

- **Forward difference** (first-order accuracy):

$$\frac{u(x_{i+1}) - u(x_i)}{h} = \frac{u_{i+1} - u_i}{h} \approx u'(x_i)$$

- **Backward difference** (first-order accuracy):

$$\frac{u(x_i) - u(x_{i-1})}{h} = \frac{u_i - u_{i-1}}{h} \approx u'(x_i)$$

- **Central difference** (second-order accuracy):

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h} = \frac{u_{i+1} - u_{i-1}}{2h} \approx u'(x_i)$$

Second-Order Derivative

The classical central difference formula for the second derivative $u''(x)$, with second-order accuracy, is:

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \approx u''(x_i)$$

2.7.3 Order of Accuracy

The order of a finite difference scheme refers to the power of h in the truncation error. Forward and backward differences are first-order accurate, while central differences are second-order accurate. Higher-order schemes can also be constructed, but this typically involves more stencil points and greater sensitivity with boundary treatments.

2.7.4 Application to Differential Equations

The finite difference method transforms a differential equation into a discrete system of algebraic equations that can be solved numerically.

For instance, consider the boundary value problem:

$$-u''(x) = f(x), \quad x \in (a, b), \quad u(a) = \alpha, \quad u(b) = \beta.$$

Using central differences, the second derivative is approximated by:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, \quad i = 1, \dots, N-1,$$

with the boundary conditions:

$$u_0 = \alpha, \quad u_N = \beta.$$

This yields a linear system that can be solved using direct methods (e.g., Gaussian elimination) or iterative solvers (e.g., Jacobi, Gauss-Seidel, SOR).

2.7.5 Additional Remarks

- **Boundary conditions:** The treatment of Dirichlet, Neumann, or Robin boundary conditions must be incorporated carefully into the scheme.
- **Stability:** Stability depends on the type of equation and the discretization parameters. For time-dependent problems, stability analysis (e.g., von Neumann analysis) is often necessary.
- **Consistency and convergence:** A scheme is consistent if its truncation error vanishes as $h \to 0$. If the scheme is also stable, then by the Lax equivalence theorem, it is convergent.
- **Higher dimensions:** The method extends naturally to two or three dimensions using regular grids and multidimensional finite difference stencils.

3

Stabilization and Observability Analysis of the Wave Equation Subject to Internal and Dynamic Conditions

This chapter widely studies the general concept of exponential stability for the wave disturbing the annular medium Ω. The important fact is that, the outer boundary Γ_1 of Ω, has a significant mass density $m > 0$ which tends to keep Γ_1 in its current state. That is, Γ_1 tries to keep moving if it was moving, and tries to rest at the rest state.

For this aim, we start by the proof of well-posedness of the model by semi-group method. Then, a result of strong stability is obtained by combining an estimate of Carleman with a principle of unique continuation. After that, the exponential stability is proved under the geometric condition (PMGC), by the method Lions-Komornik-Pruss-Liu.

This chapter is merely focused on the submitted paper (see Section 1.7).

3.1 Model (3.1)

Let $(\mathbb{R}^n, \|.\|), n \geq 2$ be a space of Banach and $\Omega \subset \mathbb{R}^n$ a bounded open domain with boundary Γ. Let Γ_0, Γ_1 be 2 subsets of Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Let ω be the non-empty damped region in Ω and covering the upper part of Ω i.e. its smooth boundary $\partial\omega = \mathcal{I} \cup \Gamma_1$ such that $\mathcal{I} \cap \Gamma_i = \emptyset$, for all $i \in \{0, 1\}$.

Now we consider the following wave equation with an internal linear damping and the mixed boundary conditions

$$\begin{cases} \theta_{tt} - \Delta\theta + d(x)\theta_t = 0 & in\ \mathbb{R}_+ \times \Omega \\ \partial_\nu\theta + m(x)\theta_{tt} - \Delta_T\theta = 0 & on\ \mathbb{R}_+ \times \Gamma_1 \\ \theta = 0 & on\ \mathbb{R}_+ \times \Gamma_0 \\ \theta(0,.) = \theta_0,\ \theta_t(0,.) = \theta_1 & in\ \Omega, \\ \theta_t(0,.) = \vartheta_0 & in\ \Gamma_1. \end{cases} \tag{3.1}$$

where $\Omega = \{x \in \mathbb{R}^n;\ 0 < r_0 < \|x\| < 1\}$ is bounded by its lower and upper parts Γ_0 and Γ_1

$$\Gamma_0 = \{x \in \mathbb{R}^n :\ \ \|x\| = r_0 < 1\} \text{ and } \Gamma_1 = \{x \in \mathbb{R}^n :\ \ \|x\| = 1\}, \tag{3.2}$$

ν the outgoing normal unit vector,
$m \in L^\infty(\Gamma_1)$ and $d \in L^\infty(\Omega)$ are non-negatives bounded functions such that

$$m(x) \geq m_0 > 0, \forall x \in \Gamma_1, \tag{3.3}$$

$$d(x) \geq d_0(constant) > 0, \quad \forall x \in \omega, \tag{3.4}$$

$$d = 0 \text{ in } \Omega - \omega. \tag{3.5}$$

Δ_T denotes the Laplace-Beltrami operator on Γ_1 satisfying:

$$\int_{\Gamma_1} v\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} \nabla_T u.\nabla_T v \mathrm{d}\Gamma_1, \quad \forall u, v \in H^1(\Gamma_1). \tag{3.6}$$

where ∇_T is the tangential gradient operator over Γ_1.
In particular,

$$\int_{\Gamma_1} u\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} |\nabla_T u|^2 \mathrm{d}\Gamma_1, \quad \forall u \in H^1(\Gamma_1).$$

We assume that $(\theta_0, \theta_1, \vartheta_0)^\top \in \mathscr{H} = \mathscr{V} \times L^2(\Omega) \times L^2(\Gamma_1)$
with

$$\mathscr{H}^1_{\Gamma_0}(\Omega) = \left\{\theta \in H^1(\Omega)\ ;\ \theta_{/\Gamma_0} = 0\right\}. \tag{3.7}$$

3.2 Semigroup Setting

This section investigates the existence and uniqueness result for the system (3.1) by using semi-groups method.
Introduce the energy space $\mathscr{H}$

$$\mathscr{H} = \mathscr{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1).$$

where $\mathscr{V}$ is the set from (3.7) equipped by the following inner product

$$\langle \theta, \theta' \rangle_{\mathscr{H}^1_{\Gamma_0}(\Omega)} = \langle \nabla\theta, \nabla\theta' \rangle_{L^2(\Omega)} + \langle \nabla_T\theta, \nabla_T\theta' \rangle_{L^2(\Gamma_1)}, \quad \forall \theta, \theta' \in \mathscr{H}^1_{\Gamma_0}(\Omega). \tag{3.8}$$

and that of Hilbert space $\mathscr{H}$

$$\langle \Theta, \Theta' \rangle_{\mathscr{H}} = \langle \theta, \theta' \rangle_{\mathscr{H}^1_{\Gamma_0}(\Omega)} + \langle \eta, \eta' \rangle_{L^2(\Omega)} + \langle m\vartheta, \vartheta' \rangle_{L^2(\Gamma_1)} \tag{3.9}$$

with $\Theta = \begin{pmatrix} \theta \\ \eta \\ \vartheta \end{pmatrix}$, $\Theta' = \begin{pmatrix} \theta' \\ \eta' \\ \vartheta' \end{pmatrix}$, and $\|.\|_{\mathscr{H}}$ denote the corresponding norm.
The energy linked to this is expressed as

$$\xi(t) = \frac{1}{2}\left[\int_\Omega \left(|\theta_t|^2 + |\nabla\theta|^2\right) \mathrm{d}x + \int_{\Gamma_1} \left(|\nabla_T\theta|^2 + m|\theta_t|^2\right) \mathrm{d}\Gamma\right] \tag{3.10}$$

The system's dissipative nature is shown through the integration by parts

$$\frac{d}{dt}\xi(t) = -\int_\Omega d|\theta_t|^2 \mathrm{d}x \tag{3.11}$$

for all regular solution $\Theta = \left(\theta, \theta_t, \theta_t/_{\Gamma_1}\right)^\top$.

An evolution system of first order on $\mathscr{H}$ whose well-posedness is equivalent to that of system (3.1) is obtained by taking $\Theta = (\theta, \theta_t, \theta_t/_{\Gamma_1})^\top$

$$\begin{cases} \Theta_t = A\Theta \\ \Theta(0) = \Theta_0 \end{cases} \tag{3.12}$$

with $\Theta_0 = (\theta_0, \theta_1, \vartheta_0)^\top$, A is the linear operator defined on $D(A) \subset \mathscr{H}$ into $\mathscr{H}$ such that

$$D(A) = \left\{\Theta = (\theta, \eta, \vartheta)^\top \in \mathscr{H} \ : \quad A\Theta \in \mathscr{H};\ \eta = \theta_t;\ \vartheta = \eta/_{\Gamma_1}\right\}$$

and for all $(\theta, \eta, \vartheta)^\top \in D(A)$

$$A(\theta, \eta, \vartheta)^\top = (\eta, \Delta\theta - d\eta, \frac{1}{m}(\Delta_T\theta - \partial_\nu\theta))^\top. \tag{3.13}$$

Lemma 3.2.1. *The operator A defined above is dissipative, i.e.,*

$$\Re e\langle A\Theta, \Theta \rangle_{\mathscr{H}} \leq 0, \quad \forall \Theta = (\theta, \eta, \vartheta)^\top \in D(A).$$

Moreover,

$$\Re e\langle A\Theta, \Theta \rangle_{\mathscr{H}} = -\int_\Omega d(x)|\eta|^2 \mathrm{d}x. \tag{3.14}$$

Proof. Let $\Theta = (\theta, \eta, \vartheta)^t \in D(A)$. Then:

$$\langle A\Theta, \Theta\rangle_{\mathscr{H}} = \left\langle \begin{pmatrix} \eta \\ \Delta\theta - d(x)\eta \\ \frac{1}{m}(\Delta_T\theta - \partial_\nu\theta) \end{pmatrix}, \begin{pmatrix} \theta \\ \eta \\ \vartheta \end{pmatrix} \right\rangle_{\mathscr{H}} .$$

Using the inner scalar product of $\mathscr{H}$ given in (3.8), we obtain:

$$\begin{aligned} \langle A\Theta, \Theta\rangle_{\mathscr{H}} &= \int_\Omega \nabla\theta \cdot \nabla\eta \,\mathrm{d}x + \int_{\Gamma_1} \nabla_T\theta \cdot \nabla_T\vartheta \,\mathrm{d}\Gamma_1 \\ &\quad + \int_\Omega \eta(\Delta\theta - d(x)\eta)\,\mathrm{d}x + \int_{\Gamma_1} m\vartheta \cdot \left(\frac{1}{m}(\Delta_T\theta - \partial_\nu\theta)\right) \mathrm{d}\Gamma_1. \end{aligned}$$

Let's denote:

$$I := \int_\Omega \eta(\Delta\theta - d(x)\eta)\,\mathrm{d}x, \quad J := \int_{\Gamma_1} \vartheta(\Delta_T\theta - \partial_\nu\theta)\,\mathrm{d}\Gamma_1.$$

Using Green's formula on I, and integrating by parts J, while noting that $\theta|_{\Gamma_0} = 0$, we find:

$$\begin{aligned} I &= -\int_\Omega \nabla\theta \cdot \nabla\eta \,\mathrm{d}x + \int_{\Gamma_1} \vartheta\partial_\nu\theta \,\mathrm{d}\Gamma_1 - \int_\Omega d(x)|\eta|^2 \,\mathrm{d}x, \\ J &= -\int_{\Gamma_1} \nabla_T\theta \cdot \nabla_T\vartheta \,\mathrm{d}\Gamma_1 - \int_{\Gamma_1} \vartheta\partial_\nu\theta \,\mathrm{d}\Gamma_1. \end{aligned}$$

Summing I and J, we obtain:

$$I + J = -\int_\Omega \nabla\theta \cdot \nabla\eta \,\mathrm{d}x - \int_{\Gamma_1} \nabla_T\theta \cdot \nabla_T\vartheta \,\mathrm{d}\Gamma_1 - \int_\Omega d(x)|\eta|^2 \,\mathrm{d}x.$$

Hence,

$$\mathfrak{Re}\langle A\Theta, \Theta\rangle_{\mathscr{H}} = -\int_\Omega d(x)|\eta|^2 \,\mathrm{d}x \leq 0,$$

which shows that A is dissipative.

The proof is complete. □

These findings enable further analysis and lead to the following result:

Theorem 3.2.2. *If $\Theta_0 \in \mathscr{H}$, then system* (3.1) *admits a unique weak solution*

$$\Theta \in C([0, +\infty), \mathscr{H}).$$

Moreover, if $\Theta_0 \in D(A)$, then system (3.1) *admits a unique strong solution*

$$\Theta \in C([0, +\infty), D(A)) \cap C^1([0, +\infty), \mathscr{H}).$$

Proof. Since A is dissipative, it suffices to verify that for all $\lambda > 0$, the operator $\lambda I - A$ is surjective. By the Lumer-Phillips theorem, this guarantees the existence and uniqueness of the solution.

To prove surjectivity, let $F = (f, g, h)^t \in \mathcal{H}$. We seek $\Theta = (u, v, v_{|_{\Gamma_1}})^t \in D(A)$, $v = u_t$ such that

$$(\lambda I - A)\Theta = F. \tag{3.15}$$

This leads to the following system:

$$\begin{cases} \lambda u - v = f, \\ (\lambda + d(x))v - \Delta u = g, \\ \lambda v_{|_{\Gamma_1}} - \dfrac{1}{m}(\Delta_T u_{|_{\Gamma_1}} - \partial_\nu u_{|_{\Gamma_1}}) = h. \end{cases} \tag{3.16}$$

From the first equation of system (3.16), we obtain

$$v = \lambda u - f, \quad \text{in } \overline{\Omega}. \tag{3.17}$$

Substituting (3.17) into the second equation of (3.16) yields:

$$a(x)u - \Delta u = f_0, \tag{3.18}$$

where $a(x) = \lambda(\lambda + d(x)) > 0$ and $f_0 = g + (\lambda + d(x))f$.

Multiplying (3.18) by $z \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ and integrating over Ω by parts, we obtain:

$$\int_\Omega a(x)uz\,dx + \int_\Omega \nabla u \cdot \nabla z\,dx - \int_{\Gamma_1} z\,\partial_\nu u\,d\Gamma_1 = \int_\Omega f_0 z\,dx. \tag{3.19}$$

From (3.17), we deduce:

$$v_{|_{\Gamma_1}} = \lambda u_{|_{\Gamma_1}} - f_{|_{\Gamma_1}}.$$

Substituting into the third equation of (3.16) gives:

$$\partial_\nu u_{|_{\Gamma_1}} = \Delta_T u_{|_{\Gamma_1}} - \lambda m u_{|_{\Gamma_1}} + mh + m\lambda f_{|_{\Gamma_1}}.$$

Injecting this into (3.19), we rewrite it as a variational formulation:

$$\varphi(u, z) = L(z), \tag{3.20}$$

where the bilinear form $\varphi(\cdot,\cdot)$ and the linear form $L(\cdot)$ are given by:

$$\begin{cases}\varphi(u,z) = \int_\Omega a(x)uz\,dx + \int_\Omega \nabla u\cdot\nabla z\,dx + \int_{\Gamma_1} \nabla_T u\cdot\nabla_T z\,d\Gamma_1 + \lambda\int_{\Gamma_1} muz\,d\Gamma_1,\\ L(z) = \int_\Omega f_0 z\,dx + \int_{\Gamma_1} mhz\,d\Gamma_1 + \lambda\int_{\Gamma_1} mfz\,d\Gamma_1.\end{cases}$$

Since $f_0 \in L^2(\Omega)$, $h, f_{|\Gamma_1} \in L^2(\Gamma_1)$, and $m \in L^\infty(\Gamma_1)$, then there exists $c > 0$ such that:

$$|L(z)| \le c\|z\|_{\mathscr{H}^1_{\Gamma_0}(\Omega)} \quad \forall z \in \mathscr{H}^1_{\Gamma_0}(\Omega).$$

So, L is continuous.

The bilinear form $\varphi(.,.)$ is continuous over $\left(\mathscr{H}^1_{\Gamma_0}(\Omega)\right)^2$ due to fact that $(a,m) \in L^\infty(\Omega)\times L^\infty(\Gamma_1)$ and the use of poincaré's inéquality (As in proof of [6, Lemma 3.2]). So, there exists a constant $M > 0$ such that:

$$|\varphi(u,z)| \le M\|u\|_{\mathscr{H}^1_{\Gamma_0}(\Omega)}\|z\|_{\mathscr{H}^1_{\Gamma_0}(\Omega)} \quad \forall u, z \in \mathscr{H}^1_{\Gamma_0}(\Omega).$$

Furthermore, φ is coercive since:

$$\varphi(u,u) \ge \|u\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)}.$$

By the Lax-Milgram theorem, problem (3.20) admits a unique solution $u \in \mathscr{H}^1_{\Gamma_0}(\Omega)$.

Now, it remains to show that $\Theta = (u, v, v_{|\Gamma_1})^t \in D(A)$. And for this, we verify the three required properties:

- Since $v = \lambda u - f$ and $\mathscr{H}^1_{\Gamma_0}(\Omega)$ is a vector space, we have $v \in \mathscr{H}^1_{\Gamma_0}(\Omega)$.
- From the second equation of (3.16), $\Delta u - d(x)v = \lambda v - g \in L^2(\Omega)$.
- From the third equation of (3.16),

$$\frac{1}{m}(\Delta_T u_{|\Gamma_1} - \partial_\nu u_{|\Gamma_1}) = \lambda v_{|\Gamma_1} - h \in L^2(\Gamma_1).$$

Thus, $\Theta \in D(A)$ satisfies (3.15), proving that A is maximal dissipative on $\mathscr{H}$. By the Lumer-Phillips theorem (see [54, Theorem 1.]), A generates a contraction semigroup $S(t)$ on $\mathscr{H}$, given by $S(t) = e^{At}$, and the solution of the Cauchy problem (3.12) is

$$\Theta(t) = S(t)\Theta_0.$$

The proof is complete. $\square$

3.3 Strong Stability result

This section analyzes the strong stability of (3.1) by applying the uniform continuation result derived from Carleman estimate and the Arendt-Batty theorem. The strong stability gives us, in addition to the monotonicity of the system's energy (3.1), the vanishing of this energy over time, which is essential for the exponential stability of (3.1). For this purpose, we consider the following elliptic operator:

$$\begin{aligned} T\ :\ H^2(\mathcal{U}) &\to L^2(\mathcal{U}) \\ \theta &\mapsto T\theta = \Delta\theta \end{aligned} \tag{3.21}$$

and the function f defined as follows

$$\begin{aligned} f\ :\ L^2(\mathcal{U}) &\to L^2(\mathcal{U}) \\ \theta &\mapsto f(\theta) = -\mu^2\theta \end{aligned} \tag{3.22}$$

The following Carleman estimate [59] is required for the sequel.

Lemma 3.3.1. *For a subset $\mathcal{U}$ of $\mathbb{R}^n$, there exist a positive constant C and a sufficiently large l_0 such that*

$$l^3\left\|e^{lZ}\theta\right\|^2_{L^2(\mathcal{U})} + l\left\|e^{lZ}\nabla_x\theta\right\|^2_{L^2(\mathcal{U})} \le C\left\|e^{lZ}\Delta\theta\right\|^2_{L^2(\mathcal{U})},\quad \forall\theta\in H^2_0(\mathcal{U}) \text{ and } l > l_0 \tag{3.23}$$

where $Z = e^{\tilde{\mu}\psi}$ with $\psi \in C^\infty(\mathbb{R}^n,\mathbb{R})$ and $\tilde{\mu} > 0$ large enough.

Now, the first result of this section can be established.

Proposition 3.3.2. *Let $\Omega \subset \mathbb{R}^n$ and $x_0 \in \Omega$. We choose the function p in a neighborhood $\mathcal{U}$ of x_0 such that $\nabla p \neq 0$ in $\overline{\mathcal{U}}$. Moreover, let $\theta \in H^2(\mathcal{U})$ be a solution of the following equation*

$$T\theta = f(\theta) \tag{3.24}$$

If $\theta = 0$ in $W = \{x \in \mathcal{U}\ :\ p(x) \ge p(x_0)\}$ then $\theta = 0$ in a neighborhood of x_0.

Proof. Let $\mathcal{U}'$ and $\mathcal{U}''$ be two neighborhoods of the point x_0 such that $\mathcal{U}'' \subset \mathcal{U}' \subset \mathcal{U}$ and $\chi \in C_c^\infty(\mathcal{U}')$ such that $\chi = 1$ in $\mathcal{U}''$ and $\tilde{\theta} = \chi\theta \in H^2_0(\mathcal{U})$. We define the functions ψ as follows

$$\psi(x) = p(x) - c|x - x_0|^2.$$

Applying Carleman estimate of lemma 3.3.1 on $\tilde{\theta}$, we get

$$l^3\int_{\mathcal{U}'} e^{2lZ}|\tilde{\theta}|^2\mathrm{d}x + l\int_{\mathcal{U}'} e^{2lZ}|\nabla\tilde{\theta}|^2\mathrm{d}x \le C\int_{\mathcal{U}'} e^{2lZ}|\Delta\tilde{\theta}|^2\mathrm{d}x \tag{3.25}$$

Since $\mathscr{U}'' \subset \mathscr{U}'$ and $\chi \in C_c^\infty(\mathscr{U}')$ with $\chi = 1$ in $\mathscr{U}''$, we obtain

$$l^3 \int_{\mathscr{U}''} e^{2lZ}|\theta|^2 \mathrm{d}x \leq C \int_{\mathscr{U}''} e^{2lZ}|\Delta\theta|^2 \mathrm{d}x + C \int_{\mathscr{U}'\backslash\mathscr{U}''} e^{2lZ}|\Delta\tilde{\theta}|^2 \mathrm{d}x$$

Thanks to (3.24): $\Delta\theta = -\mu^2\theta$, hence

$$[l^3 - \mu^4 C] \int_{\mathscr{U}''} e^{2lZ}|\theta|^2 \mathrm{d}x \leq C \int_{\mathscr{U}'\backslash\mathscr{U}''} e^{2lZ}|\Delta\tilde{\theta}|^2 \mathrm{d}x$$

then there exist $l > 0$ large enough and $C > 0$, such that

$$l^3 \int_{\mathscr{U}''} e^{2lZ}|\theta|^2 \mathrm{d}x \leq C \int_{\mathscr{U}'\backslash\mathscr{U}''} e^{2lZ}|\Delta\tilde{\theta}|^2 \mathrm{d}x \tag{3.26}$$

Since $\theta = 0$ in W, we obtain

$$l^3 \int_{\mathscr{U}''} e^{2lZ}|\theta|^2 \mathrm{d}x \leq C \int_S e^{2lZ}|\Delta\tilde{\theta}|^2 \mathrm{d}x \tag{3.27}$$

where $S = (\mathscr{U}'\backslash\mathscr{U}'') \cup W$. Let us set $\phi_\epsilon^1 = \{x \in \mathscr{U} \ : \ Z(x) \leq Z(x_0) - \epsilon\}$
and $\phi_\epsilon^2 = \{x \in \mathscr{U} \ : \ Z(x) \geq Z(x_0) - \epsilon\}$. Suppose there exists $\epsilon > 0$ such that $S \subset \phi_\epsilon = \phi_\epsilon^1 \cap \phi_\epsilon^2$. In this case, choose a ball B_0 centered at x_0 where $B_0 \subset \mathscr{U}'' \cap \phi_{\frac{\epsilon}{2}}^2$. Using (3.27), we get

$$\int_{B_0} |\theta|^2 \mathrm{d}x \leq \frac{Ce^{-\epsilon l}}{l^3} \int_S |\Delta\tilde{\theta}|^2 \mathrm{d}x$$

When $l \to +\infty$, we get $\theta = 0$ in B_0. □

Theorem 3.3.3. *(Calderón theorem) Assume that $\omega \subset \Omega \subset \mathbb{R}^n$, with $\omega \neq \emptyset$. If $\theta \in H^2(\Omega)$ such that $T\theta = f(\theta)$ in Ω and $\theta = 0$ in ω then θ vanishes in Ω.*

Proof. Let $F = supp\, \theta$, we get the desired result by using [59]. □

Theorem 3.3.4. *The semi-group $(e^{tA})_{t\geq 0}$ is strongly stable in the energy space $\mathscr{H}$ if*

$$\lim_{t\to+\infty} \|e^{tA}\Theta_0\|_{\mathscr{H}} = 0, \quad \forall\Theta_0 \in \mathscr{H}.$$

Proof. Since A is compact in $\mathscr{H}$, the system (3.1) is strongly stable if and only if A does not possess pure imaginary eigenvalues i.e $\sigma(A) \cap i\mathbb{R} = \emptyset$.
From Section 2, we know that, we have $0 \in \rho(A)$. It is now necessary to demonstrate that $\sigma(A) \cap i\mathbb{R}^* = \emptyset$. To establish this, we will proceed by contradiction. We suppose that there exist $\mu \neq 0$ and $\Theta = (\theta, \eta, \vartheta)^t \in D(A)\backslash\{0\}$ such that

$$A\Theta = i\mu\Theta. \tag{3.28}$$

which give us,

$$\eta = i\mu\theta, \quad \text{in } \Omega \tag{3.29}$$

$$\Delta\theta - d\eta = i\mu\eta, \quad \text{in } \Omega \tag{3.30}$$

$$\Delta_T\theta - \partial_\nu\theta = i\mu m\vartheta, \quad \text{on } \Gamma_1 \tag{3.31}$$

moreover,

$$\theta = 0 \quad \text{on } \Gamma_0 \tag{3.32}$$

$$\vartheta = \eta_{/\Gamma_1} \tag{3.33}$$

Using (3.14) and (3.28), we get

$$0 = \Re e(i\mu\|\Theta\|_H^2) = \Re e(\langle A\Theta, \Theta\rangle) = -\int_\Omega d|\eta|^2 \mathrm{d}x \leq 0. \tag{3.34}$$

Thus, from (3.29) and (3.34), we get

$$d\theta = 0, \text{ in } \Omega \quad \text{and consequently } \theta = \eta = 0 \text{ in } \omega \tag{3.35}$$

Combining (3.29), (3.30) and (3.35), we get

$$\Delta\theta + \mu^2\theta = 0 \text{ in } \Omega \tag{3.36}$$

And then combining (3.29), (3.31) and (3.33), we obtain

$$\partial_\nu\theta = \Delta_T\theta + \mu^2 m\theta \text{ on } \Gamma_1 \tag{3.37}$$

So we get the following system

$$\begin{cases} \Delta\theta = -\mu^2\theta \text{ in } \Omega \\ \partial_\nu\theta = \Delta_T\theta + \mu^2 m\theta \text{ on } \Gamma_1 \\ \theta = \eta = 0 \text{ in } \omega \end{cases} \tag{3.38}$$

Using the first and third equation of the system (3.38) with the theorem 3.3.3, we get

$$\theta = \eta = 0 \text{ in } \Omega. \tag{3.39}$$

Now, it remains to show that $\theta_{|\Gamma_1} = 0$.

For this, we first recall that the trace operator $Tr : H^2(\Omega) \to H^{\frac{3}{2}}(\Gamma)$ is continuous. That is, there exists a constant $c_0 > 0$ such that:

$$\|\theta\|_{H^{\frac{3}{2}}(\Gamma)} \leq c_0 \|\theta\|_{H^2(\Omega)}.$$

Since $H^{\frac{3}{2}}(\Gamma) \hookrightarrow H^1(\Gamma)$ and $\theta_{/\Gamma_0} = 0$, then there exists $c_1 > 0$ such that

$$\|\nabla_T \theta\|_{L^2(\Gamma_1)} \leq c_1 \|\theta\|_{H^2(\Omega)}. \tag{3.40}$$

(a) On one hand, multiplying $(3.38)_1$ by $\bar{\theta}$ and then integrating over Ω, it follows that:

$$\int_\Omega \bar{\theta} \Delta \theta \mathrm{d}x = -\mu^2 \int_\Omega |\theta|^2 \mathrm{d}x$$

Applying Green's formula with $(3.38)_2$ on the left hand side, we get

$$\|\theta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)} := \int_\Omega |\nabla \theta|^2 \mathrm{d}x + \int_{\Gamma_1} |\nabla_T \theta|^2 \mathrm{d}\Gamma_1 = \mu^2 \left(\int_\Omega |\theta|^2 \mathrm{d}x + \int_{\Gamma_1} m|\theta|^2 \mathrm{d}\Gamma_1 \right) \tag{3.41}$$

(b) On the other hand, multiplying $(3.38)_1$ by $\Delta\bar{\theta}$, we get

$$\int_\Omega |\Delta \theta|^2 \mathrm{d}x = -\mu^2 \int_\Omega \theta \Delta \bar{\theta} \mathrm{d}x.$$

Applying Green's formula with $(3.38)_2$ on the right hand side, we get

$$\int_\Omega |\Delta \theta|^2 \mathrm{d}x = \mu^2 \left(\int_\Omega |\nabla \theta|^2 \mathrm{d}x + \int_{\Gamma_1} |\nabla_T \theta|^2 \mathrm{d}\Gamma_1 \right) - \mu^4 \int_{\Gamma_1} m|\theta|^2 \mathrm{d}\Gamma_1. \tag{3.42}$$

Combining (3.41) and (3.42), it follows that

$$\|\theta\|^2_{H^2(\Omega)} := \int_\Omega |\Delta \theta|^2 \mathrm{d}x + \int_\Omega |\nabla \theta|^2 \mathrm{d}x + \int_\Omega |\theta|^2 \mathrm{d}x = \mu^4 \int_\Omega |\theta|^2 \mathrm{d}x + \int_\Omega |\nabla \theta|^2 \mathrm{d}x + \int_\Omega |\theta|^2 \mathrm{d}x.$$

Using (3.39), it follows that $\theta = 0$ in $H^2(\Omega)$. Then the result (3.40) gives $\nabla_T \theta = 0$ in $L^2(\Gamma_1)$. Thus using that with (3.39) in (3.41), it follows that $\theta = 0$ over Γ_1. Consequently, $\Theta = 0$ which is a contradiction with the fact that $\Theta \in D(A) \backslash \{0\}$.

The proof is complete. □

3.4 Analysis of Exponential Stability

In this section, we address the exponential stability of the system referred to by (3.1). To facilitate this analysis, we implement a geometric condition named "Piecewise Multiplier Geometric Condition" (referenced in [3, Definition 4.1]) on the subset $\omega \subset \Omega$, which is essential for ensuring effective dissipation. Our approach is inspired by established methodologies as described by [67], and is further supported by the studies of [65] and [49].

Definition 3.4.1. *(PMGC) Let $(x_j)_{j\in\mathbb{N}^*}$ be sequence of point in $\mathbb{R}^n$, $n \geq 2$. Assume that there exists $(\Omega_j)_{j\in\mathbb{N}^*}$, a sequence of subsets in Ω with smooth boundary $\Gamma_j^0 = \partial\Omega_j$ such that $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \mathbb{N}^*$, $i < j$. We say that, ω satisfies the PMGC if and only if*

$$\Omega \cap \mathcal{N}_\epsilon\left[\left(\bigcup_{j=1}^{J}\gamma_j\right) \cup \left(\Omega\backslash\bigcup_{j=1}^{J}\Omega_j\right)\right] \subset \omega$$

for some $\epsilon > 0$.
Here, $\mathcal{N}_\epsilon$ and γ_j are defined as follows:

$$\mathcal{N}_\epsilon(S) = \bigcup_{x\in S}\{y \in \mathbb{R}^n \ : \ |x-y| < \epsilon\}, \quad \gamma_j = \{x \in \Gamma_j^0 \ : \ \vec{x_j x}\,.\,\nu_j(x) > 0\}, \quad \vec{x_j x} = x - x_j.$$

for $S \subset \mathbb{R}^n$ and ν_j is the unit normal vector pointing into the exterior of Ω_j.
$\vec{x_j x}$ is the vector from x_j toward x.

To align this definition with the specifics of our study, the introduction of additional notations is necessary. Set $m_j(x) = x - x_j$ and $R_j = sup\{\|m_j(x)\|_2, x \in \Omega\}$, for all $j \in 1, \dots, J$.
Let $0 < \epsilon_0' < \epsilon_0 < \epsilon_1 < \epsilon$, since

$$\left(\overline{\Omega}\backslash\mathcal{N}_{\epsilon_1}\right) \cap \overline{\mathcal{N}_{\epsilon_0}} = \emptyset \text{ and } \left(\overline{\Omega}\backslash\mathcal{N}_{\epsilon_0}\right) \cap \overline{\mathcal{N}_{\epsilon_0'}} = \emptyset.$$

Set

$$\mathcal{S} = \left(\bigcup_{j=1}^{J}\gamma_j\right) \cup \left(\Omega\backslash\bigcup_{j=1}^{J}\Omega_j\right), \quad \mathcal{O}_0' = \mathcal{N}_{\epsilon_0'}(S), \quad \mathcal{O}_k = \mathcal{N}_{\epsilon_k}(S), \quad k \in \{0,1\},$$

. The following embedding property is thus obvious:

$$\mathcal{O}_0' \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \omega \tag{3.43}$$

Now, let's introduce two cut-off function $\varphi \in W^{1,\infty}(\Omega)$, such that $0 \leq \varphi \leq 1$ and

$$\varphi = \begin{cases} 1 : x \in \mathcal{O}_0 \\ 0 : x \in \Omega\backslash\mathcal{O}_1 \end{cases}$$

Since $\left(\overline{\Omega}\backslash\mathcal{O}_1\right) \cap \overline{\mathcal{O}_0} = \emptyset$, now we can define the function $\psi_j \in W^{1,\infty}(\Omega)$, $0 \leq \psi_j \leq 1$, and we consider the $\mathscr{C}^1$ vector field σ_j on Ω_j, such that

$$\psi_j = \begin{cases} 1 \text{ if } x \in \overline{\Omega_j}\backslash\mathcal{O}_0 \\ 0 \text{ if } x \in \mathcal{O}_0' \end{cases} \text{ and } \sigma_j = \psi_j m_j.$$

We denote $\Gamma_j^c = (\Gamma_j^0\backslash\gamma_j) \cap \Gamma$, $\gamma_j^c = \Gamma_j^0\backslash(\Gamma_j^c \cup \gamma_j)$. See the following figure:

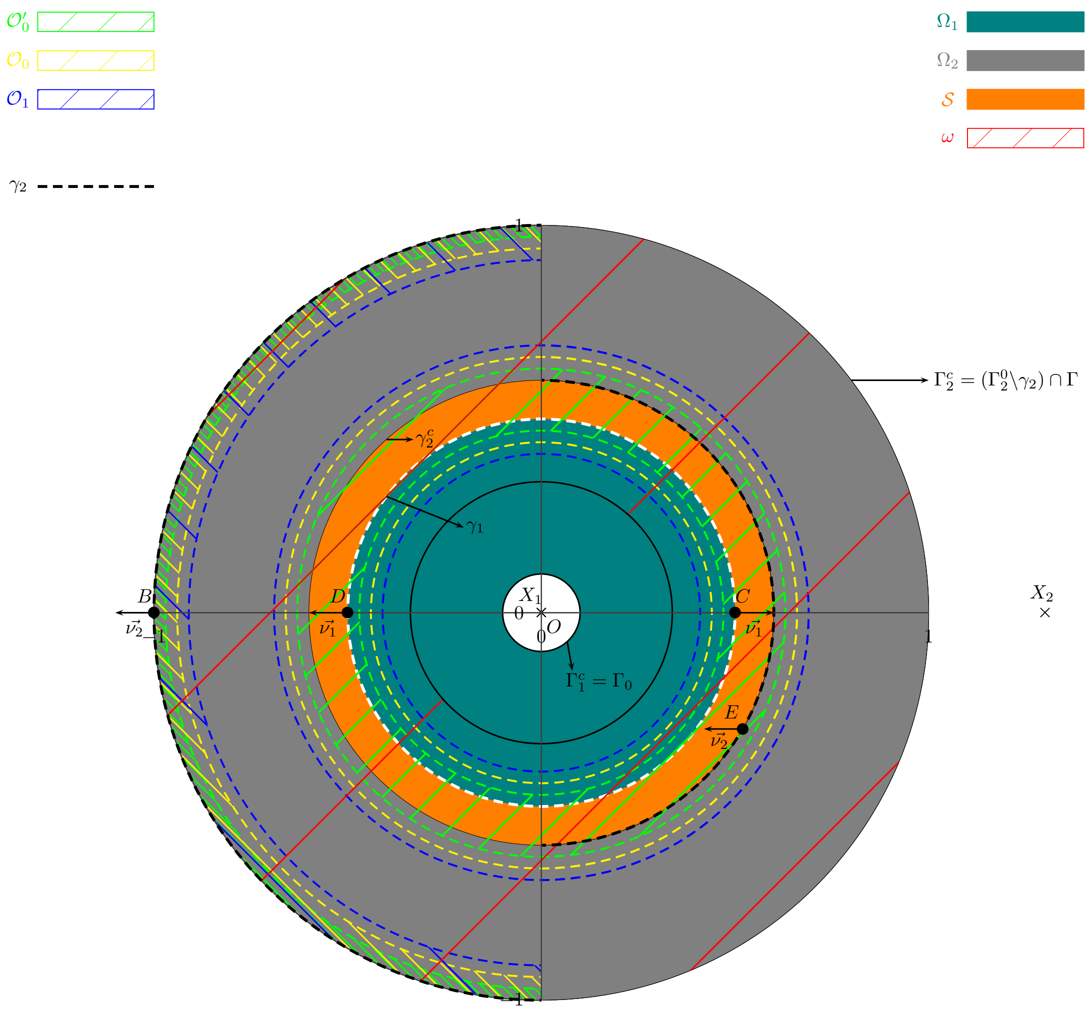

Figure 1 Geometric description (**PMGC**) of the domain Ω for $n = J = 2$.

Important 3.4.2. *In order to have a readable figure, parts $\mathcal{O}_k, k \in \{0, 1\}$ have not been entirely hatched by their respective colors except $\mathcal{O}'_0$ in green. We consider inclusions* (3.43) *in order to locate the entire $\mathcal{O}_i$ domains $i \in \{0, 1\}$. Moreover, by the definition of γ_j, S and $\mathcal{O}$ it follows that the sets $\Gamma_1 \cap \mathcal{O}'_0$ and $\Gamma_1 \cap \mathcal{O}_k$ are non-empty such that:*

$$\Gamma_1 \cap \mathcal{O}'_0 = \Gamma_1 \cap \mathcal{O}_k = \gamma_2 \cap \Gamma_1.$$

We are now in position to state the exponential stability of the system (3.1).

Theorem 3.4.3. *Assume that ω satisfies the **PMGC** condition.*
The $\mathscr{C}_0$ semi-group of contraction $\left(e^{tA}\right)_{t\geq 0}$ is exponentially stable if there exist $M \geq 1$ and $\sigma > 0$ independent of Θ_0 such that

$$\left\|e^{tA}\Theta_0\right\|_{\mathscr{H}} \leq Me^{-\sigma t}\left\|\Theta_0\right\|_{\mathscr{H}} \tag{3.44}$$

According to [44] and [81], a $\mathscr{C}_0$ semi-group of contraction $\left(e^{tA}\right)_{t\geq 0}$ on $\mathscr{H}$ satisfies (3.44) if

$$i\mathbb{R} \subset \rho(A) \tag{3.45}$$

$$\sup_{\mu\in\mathbb{R}}\left\|(i\mu I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} = o(1). \tag{3.46}$$

Since $i\mathbb{R} \subset \rho(A)$ (*see* **theorem** 3.3.4 then the condition (3.45) is satisfied. And concerning the condition (3.46), its proof will be given by a proof by contradiction. For this aim, we suppose that the condition (3.46) is false, then there exists a sequence $(\mu_n, \Theta_n)_{n\geq 1} \subset \mathbb{R}^* \times D(A)$ with

$$\lim_{n\to+\infty}|\mu_n| = +\infty \text{ and } \|\Theta_n\|_{\mathscr{H}} = \left\|(\theta_n, \eta_n, w_n)^t\right\|_{\mathscr{H}} = 1 \tag{3.47}$$

such that

$$(i\mu_n I - A)\Theta_n = F_n = (f_n, g_n, h_n)^t \to 0 \text{ in } \mathscr{H}. \tag{3.48}$$

To simplify, we omit the index n. (3.48) gives us the following system

$$\begin{cases} i\mu\theta - \eta = f \\ i\mu\eta - \Delta\theta + d\eta = g \\ i\mu m\vartheta - \Delta_T\theta + \partial_\nu\theta = mh \end{cases} \tag{3.49}$$

By inserting $(3.49)_1$ in $(3.49)_2$ and $(3.49)_3$ and knowing that $\vartheta = \eta_{/\Gamma_1}$, we get

$$\Delta\theta + \mu^2\theta - i\mu d\theta = F_1 \tag{3.50}$$

$$\partial_\nu\theta - \mu^2 m\theta - \Delta_T\theta = G_1 \tag{3.51}$$

where,

$$F_1 = -g - (d + i\mu)f \quad \text{ and } \quad G_1 = m(h + i\mu f) \tag{3.52}$$

Before continuing, let's see the monotonicity of the solution $(\theta, \eta, \vartheta)^t$ of the system (3.1).

Lemma 3.4.4. *The solution* $(\theta, \eta, \vartheta)^t$ *of problem* (3.1) *verify the estimates*

$$\int_\Omega d|\eta|^2 \mathrm{d}x = o(1), \quad \int_\Omega d|\mu\theta|^2 \mathrm{d}x = o(1), \quad \int_\omega |\mu\theta|^2 \mathrm{d}x = o(1), \quad \textit{and} \quad \int_{\Gamma_1 \cap \mathcal{O}_0} |\mu\theta|^2 \mathrm{d}x = o(1) \tag{3.53}$$

Proof. • We consider the inner product of the equation (3.48) with Θ in $\mathcal{H}$, using the fact that $\|\Theta\|_{\mathcal{H}} = 1$ and $\|F\|_{\mathcal{H}} = o(1)$, we obtain

$$\int_\Omega d|\eta|^2 \mathrm{d}x = -\Re e(\langle A\Theta, \Theta\rangle_{\mathcal{H}}) = \Re e(\langle (i\mu I - A)\Theta, \Theta\rangle_{\mathcal{H}}) = \Re e(\langle F, \Theta\rangle_{\mathcal{H}}) = o(1) \tag{3.54}$$

- Using $(3.49)_1$ and the fact that $\|F\|_{\mathcal{H}} = o(1)$, we obtain the second estimation of (3.53).

- The 3^{rd} estimation is obvious by the definition of d.

- Suppose for $x_0 \in \mathbb{R}^n$, $l(x) = x - x_0$ over $\overline{\Omega}$ such that

$$l\,.\,\nu \geq \epsilon > 0, \forall x \in \gamma_j \cap \Gamma_1 \quad \text{and} \quad l\,.\,\nu \leq 0, \ \forall x \in \Gamma_0. \tag{3.55}$$

Set

$$R_0 = \sup_{x \in \overline{\Omega}}\{\|l(x)\|_2\} < +\infty, \quad I_e = 2\mu^2 \int_\Omega \varphi\theta(l\,.\,\nabla\theta)\mathrm{d}x,$$

where (.) is the standard scalar product with $\|.\|_2$ as its norm.

On one hand, we have by Cauchy-Schwarz's inequality:

$$|I_e| \leq 2R_0 \left(\int_\Omega \varphi|\mu\nabla\theta|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \varphi|\mu\theta|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

Since $\mu\nabla\theta$ is uniformly bounded in $L^2(\Omega)$ then using $(3.53)_3$, it follows that $|I_e| = o(1)$. Which implies that $I_e = o(1)$.

On another hand, applying an integration by parts, we get

$$I_e = -2\mu^2 \int_\Omega div(l\varphi\theta)\theta \mathrm{d}x + 2\mu^2 \int_{\Gamma_1} (l\,.\,\nu)\varphi|\theta|^2 \mathrm{d}\Gamma_1,$$

Knowing that $div(\vec{f}g) = g\,div\vec{f} + \vec{f}.\nabla g$ and $div(l) = n$, it follows that:

$$\begin{aligned} I_e &= -2n\mu^2 \int_\Omega \varphi|\theta|^2 \mathrm{d}x - 2\mu^2 \int_\Omega (l\,.\,\nabla\varphi)|\theta|^2 \mathrm{d}x - I_e + 2\mu^2 \int_{\Gamma_1} (l\,.\,\nu)\varphi|\theta|^2 \mathrm{d}\Gamma_1 \\ &= -2n \int_\Omega \varphi|\mu\theta|^2 \mathrm{d}x - 2\int_\Omega (l\,.\,\nabla\varphi)|\mu\theta|^2 \mathrm{d}x - I_e + 2\int_{\Gamma_1} (l\,.\,\nu)\varphi|\mu\theta|^2 \mathrm{d}\Gamma_1 \end{aligned}$$

Consequently,

$$\int_{\Gamma_1}(l.\nu)\varphi|\mu\theta|^2 \mathrm{d}\Gamma_1 = \underbrace{I_e}_{=o(1)} + n\int_\Omega \varphi|\mu\theta|^2\mathrm{d}x + \int_\Omega (l.\nabla\varphi)|\mu\theta|^2\mathrm{d}x.$$

Now, using the definition of φ with $(3.53)_3$, we get:

$$\int_{\Gamma_1\cap\mathcal{O}_0}(l.\nu)|\mu\theta|^2\mathrm{d}\Gamma_1 = o(1)$$

Finally, using (3.55), we get

$$\epsilon\int_{\Gamma_1\cap\mathcal{O}_0}|\mu\theta|^2\mathrm{d}\Gamma_1 \leq \int_{\Gamma_1\cap\mathcal{O}_0} l.\nu|\mu\theta|^2\mathrm{d}\Gamma_1 = o(1)\to 0. \tag{3.56}$$

Hence, $\int_{\Gamma_1\cap\mathcal{O}_0}|\mu\theta|^2\mathrm{d}\Gamma_1 = o(1)$.

The proof is complete. $\square$

Remark 3.4.5. *In the proof of estimation* $(3.53)_4$*, it is easily verifiable that*

$$\int_{\Gamma_1}|\mu\theta|^2\mathrm{d}\Gamma_1 = o(1),$$

by simply considering the cut-off function φ *such that*

$$\varphi = \begin{cases} 1, & \text{in } \omega\cap V \\ 0, & \text{otherwise} \end{cases}$$

where V *is a neighborhood of* Γ_j^c *in* ω*. Then this do not impact on the geometric condition (PMGC) when* l *is chosen as*

$$l.\nu \leq 0, \quad \text{over } \Gamma_j^c.$$

Lemma 3.4.6. *For any solution* $(\theta,\eta,\vartheta)^t$ *of* (3.1)*, the following estimates*

$$\int_{\Omega\cap\mathcal{O}_0}|\nabla\theta|^2\mathrm{d}x = o(1), \quad \text{and} \int_{\Gamma_1\cap\mathcal{O}_0}|\nabla_T\theta|^2\mathrm{d}\Gamma_1 = o(1). \tag{3.57}$$

hold true

Proof. Multiplying (3.50) by $-\varphi\bar{\theta}$ and integrating over Ω and then taking the real parts, we get

$$-\Re e\left(\int_\Omega \varphi\bar{\theta}\Delta\theta\mathrm{d}x\right) - \int_\Omega\varphi|\mu\theta|^2\mathrm{d}x + \underbrace{\Re e\left(i\mu\int_\Omega\varphi d|\theta|^2\mathrm{d}x\right)}_{=0} = -\Re e\left(\int_\Omega\varphi\bar{\theta}F_1\mathrm{d}x\right)$$

Then, using the definition of φ and $(3.53)_3$, we get

$$-\Re e\left(\int_\Omega\varphi\bar{\theta}\Delta\theta\mathrm{d}x\right) + \Re e\left(\int_\Omega\varphi\bar{\theta}F_1\mathrm{d}x\right) = o(1) \tag{3.58}$$

Now, set

$$R_1 = \Re e\left(\int_\Omega\Delta\theta\varphi\bar{\theta}\mathrm{d}x\right), \quad R_2 = \Re e\left(\int_\Omega\varphi F_1\bar{\theta}\mathrm{d}x\right).$$

- Applying Green's formula to the integral of R_1 and using the condition $\theta = 0$ on Γ_0, we get

$$R_1 = -\Re e\left(\int_\Omega \nabla(\varphi\bar{\theta}) . \nabla\theta \mathrm{d}x\right) + \Re e\left(\int_{\Gamma_1} \varphi\bar{\theta}\partial_\nu\theta \mathrm{d}\Gamma_1\right) \tag{3.59}$$

Using (3.51), we get

$$\begin{aligned} R_1 = &-\Re e\left(\int_\Omega \bar{\theta}\nabla\theta\nabla\varphi \mathrm{d}x\right) - \int_\Omega \varphi|\nabla\theta|^2 \mathrm{d}x + \Re e\left(\int_{\Gamma_1} G_1\varphi\bar{\theta}\mathrm{d}\Gamma_1\right) \\ &- \int_{\Gamma_1} \varphi|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 - \Re e\left(\int_{\Gamma_1} \bar{\theta}\nabla_T\theta\nabla_T\varphi \mathrm{d}\Gamma_1\right) + \int_{\Gamma_1} m\varphi|\mu\theta|^2 \mathrm{d}\Gamma_1 \end{aligned} \tag{3.60}$$

By utilizing the definition of φ and knowing that $\nabla\theta$ (*resp.* $\nabla_T\theta$) is uniformly bounded in $L^2(\Omega)$ (*resp.* $L^2(\Gamma_1)$), and taking into account the estimations $\|F\|_H = o(1)$, we derive the following:

$$\Re e\left(\int_\Omega \bar{\theta}\nabla\theta\nabla\varphi \mathrm{d}x\right) = o(\mu^{-1}), \quad \Re e\left(\int_{\Gamma_1} \bar{\theta}\nabla_T\theta\nabla_T\varphi \mathrm{d}\Gamma_1\right) = o(1), \quad \Re e\left(\int_{\Gamma_1} G_1\varphi\bar{\theta}\mathrm{d}\Gamma_1\right) = o(1).$$

By considering $(3.53)_4$ and the boundedness of m in $L^\infty(\Omega)$, we therefore get:

$$R_1 = o(1) - \int_\Omega \varphi|\nabla\theta|^2 \mathrm{d}x - \int_{\Gamma_1} \varphi|\nabla_T\theta|^2 \mathrm{d}\Gamma_1. \tag{3.61}$$

- It can be easily seen that $R_2 = o(1)$ by the use of the definition of φ and the fact that $\|F\|_{\mathscr{H}} = o(1)$.

Consequently, the main equation (3.58) satisfies:

$$o(1) - \int_\Omega \varphi|\nabla\theta|^2 \mathrm{d}x - \int_{\Gamma_1} \varphi|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 = o(1)$$

which implies,

$$\int_\Omega \varphi|\nabla\theta|^2 \mathrm{d}x = o(1), \quad \int_{\Gamma_1} \varphi|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 = o(1).$$

Applying the definition of φ, we get the desired result.
The proof is complete. □

Remark 3.4.7. *Considering the cut-off function $\varphi = 1$ over ω and 0 otherwise, it is easily verifiable that*

$$\int_\omega |\nabla\theta|^2 \mathrm{d}x = o(1), \quad \int_{\Gamma_1} |\nabla_T\theta|^2 \mathrm{d}\Gamma_1 = o(1).$$

Lemma 3.4.8. *For any solution* $(\theta, \eta, \vartheta)^t$ *of* (3.1)*, the following result remains true:*

$$2\int_{\Gamma_j^0\cap\Gamma_1}\left(\sigma_j.\nabla\bar{\theta}-\sigma_j^T.\nabla_T\bar{\theta}\right)\partial_{\nu_j}\theta d\Gamma_1 = 2\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j|\partial_{\nu_j}\theta|^2 d\Gamma_1 \tag{3.62}$$

where σ_j^T *denotes the tangential component of* σ_j *and* $(.) = \langle ., . \rangle$ *as the standard scalar product of the euclidean space* $\mathbb{R}^n$.
In addition,

$$|\nabla\theta|^2 = |\nabla_T\theta|^2 + |\partial_{\nu_j}\theta|^2, \quad over\ \Gamma_j^0\cap\Gamma_1. \tag{3.63}$$

Proof. By definition, the tangential component $\vec{v}_T$ of a vector $\vec{v}\in\mathbb{C}^n$:

$$\vec{v}_T = \vec{v} - \langle\vec{v},\nu\rangle\nu.$$

Set $a_j := \langle\sigma_j,\nu_j\rangle\in\mathbb{R}$ and $b_j := \partial_{\nu_j}\bar{\theta} = \langle\nabla\bar{\theta},\nu_j\rangle\in\mathbb{C}$.
We thus get,

$$\sigma_j.\nabla\bar{\theta}-\sigma_j^T.\nabla_T\bar{\theta} = \langle\sigma_j,\nabla\bar{\theta}\rangle - \langle\sigma_j - a_j\nu_j, \nabla\bar{\theta} - b_j\nu_j\rangle$$

Using linear property of $\langle ., . \rangle$ and the fact that $a_j\in\mathbb{R}$ and $b_j\in\mathbb{R}$, it follows that:

$$\begin{aligned}\sigma_j.\nabla\bar{\theta}-\sigma_j^T.\nabla_T\bar{\theta} &= a_j\langle\nu_j,\nabla\bar{\theta}\rangle + \overline{b_j}\langle\sigma_j,\nu_j\rangle - a_j\overline{b_j}\underbrace{\langle\nu_j,\nu_j\rangle}_{=1}\\ &= a_j\overline{b_j} + \overline{b_j}a_j - a_j\overline{b_j}\\ &= a_j\overline{b_j}.\end{aligned}$$

Then,

$$\begin{aligned}\sigma_j.\nabla\bar{\theta}-\sigma_j^T.\nabla_T\bar{\theta} &= \langle\sigma_j,\nu_j\rangle\overline{\langle\nabla\bar{\theta},\nu_j\rangle}\\ &= \sigma_j.\nu_j\partial_{\nu_j}\theta.\end{aligned}$$

Finally, it follows that:

$$\begin{aligned}2\int_{\Gamma_j^0\cap\Gamma_1}\left(\sigma_j.\nabla\bar{\theta}-\sigma_j^T.\nabla_T\bar{\theta}\right)\partial_{\nu_j}\theta d\Gamma_1 &= 2\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j\partial_{\nu_j}\bar{\theta}\partial_{\nu_j}\theta d\Gamma_1\\ &= 2\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j|\partial_{\nu_j}\theta|^2 d\Gamma_1.\end{aligned}$$

The proof of the additional (3.63) is obvious from the definitions of $\partial_{\nu_j}\theta$ and $\nabla_T\theta$. Which is:

$$\partial_{\nu_j}\theta = \langle\nabla\theta,\nu_j\rangle, \quad \nabla_T\theta = \nabla\theta - \partial_{\nu_j}\theta\nu_j.$$

The proof is complete. □

Lemma 3.4.9. *For any solution* $(\theta, \eta, \vartheta)^t$ *of* (3.1)*, the following estimate*

$$n\int_{\Pi}|\mu\theta|^2 \mathrm{d}\Pi - (n-2)\int_{\Pi}|\nabla\theta|^2\mathrm{d}\Pi + (n-1)\int_{\Sigma} m|\mu\theta|^2\mathrm{d}\Sigma - (n-3)\int_{\Sigma}|\nabla_T\theta|^2\mathrm{d}\Sigma \leq o(1). \quad (3.64)$$

hold true. where $\Pi = \Omega\backslash(\Omega\cap\mathcal{O}_0)$ *and* $\Sigma = \bigcup_{j=1}^{J}(\Gamma_j^c \cap \Gamma_1)$.

Proof. First of all, keep in mind that the definition of σ_j implies that:

$$\begin{cases} \nabla\sigma_j = \psi_j \nabla m_j = \psi_j I_n \\ \mathrm{div}(\sigma_j) = n\psi_j. \end{cases}$$

where I_n is the identity vector in $\mathbb{R}^n$ and $n = dim(\Omega)$.
Moreover, the tangential component of σ_j denoted by σ_j^T satisfies

$$\begin{cases} \sigma_j^T = \sigma_j - \sigma_j.\nu_j\nu_j \\ \nabla_T\sigma_j^T = \psi_j\nabla_T m_j^T = \psi_j I_{n-1} - \sigma_j.\nu_j B \\ \mathrm{div}_T(\sigma_j^T) = (n-1)\psi_j - \sigma_j.\nu_j Tr(B), \end{cases}$$

where $(.) = \langle , \rangle$ is the standard euclidean scalar product,
B is the second fondamental form of Γ (curvature operator over Γ_j^0) and $Tr(B)$ its trace (see [47]).

- **First step**

  Let's start by showing this estimate

$$\begin{aligned} &n\int_{\Omega_j}\psi_j|\mu\theta|^2\,\mathrm{d}x - (n-2)\int_{\Omega_j}\psi_j|\nabla\theta|^2\,\mathrm{d}x \\ &-(n-3)\int_{\Gamma_j^c\cap\Gamma_1}|\nabla_T\theta|^2\,\mathrm{d}\Gamma_1 + (n-1)\int_{\Gamma_j^c\cap\Gamma_1} m|\mu\theta|^2\,\mathrm{d}\Gamma_1 \leq o(1). \end{aligned} \quad (3.65)$$

  To do this, we proceed as follows.

I. Multiplying (3.50) by $(2\sigma_j.\nabla\bar{\theta})$ and integrating over Ω_j and taking the real parts, we get

$$\begin{aligned} &\Re e\left(2\int_{\Omega_j}(\sigma_j.\nabla\bar{\theta})\Delta\theta\,\mathrm{d}x\right) + \Re e\left(2\mu^2\int_{\Omega_j}(\sigma_j.\nabla\bar{\theta})\theta\,\mathrm{d}x\right) \\ &-\Re e\left(2i\mu\int_{\Omega_j}d(x)(\sigma_j.\nabla\theta)\theta\,\mathrm{d}x\right) = \Re e\left(2\int_{\Omega_j}F_1(\sigma_j.\nabla\bar{\theta})\,\mathrm{d}x\right) \end{aligned} \quad (3.66)$$

- First by using (3.52), we get

$$2\int_{\Omega_j}F_1(\sigma_j.\nabla\bar{\theta})\,\mathrm{d}x = -2\int_{\Omega_j}(g+f)\sigma_j.\nabla\bar{\theta}\,\mathrm{d}x - 2i\mu\int_{\Omega_j}d(x)f(\sigma_j.\nabla\bar{\theta})\,\mathrm{d}x \quad (3.67)$$

Knowing that $\mu\theta$ and $\nabla\theta$ are uniformly bounded in $L^2(\Omega)$, f and g tend to 0 in $\mathscr{H}$ and as σ_j and d are bounded on Ω_j, we deduce that

$$\Re e\left(2\int_{\Omega_j} F_1(\sigma_j.\nabla\bar{\theta})\,\mathrm{d}x\right) = o(1) \tag{3.68}$$

In addition, by using the definition of d and $(3.53)_2$ with the uniform boundedness of $\nabla\theta$ and the boundedness of σ_j in $\overline{\Omega}$, we also deduce that:

$$\Re e\left(2i\mu\int_{\Omega_j} d(x)(\sigma_j.\nabla\theta)\theta\,\mathrm{d}x\right) = o(1) \tag{3.69}$$

Now, set I_i and R_i as follows:

$$I_1 = 2\int_{\Omega_j}(\sigma_j.\nabla\bar{\theta})\Delta\theta\,\mathrm{d}x, \quad I_2 = 2\mu^2\int_{\Omega_j}(\sigma_j.\nabla\bar{\theta})\theta\,\mathrm{d}x, \quad R_i = \Re e(I_i), i\in\{1,2\}.$$

- **For** R_1

Applying Green's formula on I_1, we obtain

$$I_1 = -2\int_{\Omega_j}\nabla(\sigma_j.\nabla\bar{\theta}).\nabla\theta\mathrm{d}x + 2\int_{\Gamma_j^0}(\sigma_j.\nabla\bar{\theta})\partial_\nu\theta\mathrm{d}\Gamma_j^0.$$

Since, we have:

$$\begin{aligned}\nabla(\sigma_j.\nabla\bar{\theta}).\nabla\theta &= \nabla\bar{\theta}.\nabla\sigma_j.\nabla\theta + \sigma_j.\nabla\theta\nabla(\nabla\bar{\theta})\\ &= \nabla\bar{\theta}.\nabla\sigma_j.\nabla\theta + \frac{1}{2}\sigma_j.\nabla(|\nabla\theta|^2),\end{aligned}$$

then

$$\begin{aligned}I_1 =& -2\int_{\Omega_j}\nabla\bar{\theta}.\nabla\sigma_j.\nabla\theta\mathrm{d}x - \int_{\Omega_j}\sigma_j.\nabla(|\nabla\theta|^2)\mathrm{d}x\\ &+\underbrace{2\int_{\Gamma_j^0}(\sigma_j.\nabla\bar{\theta})\partial_{\nu_j}\theta\mathrm{d}\Gamma_j^0}_{:=J_1}\\ \\ =& -2\int_{\Omega_j}\nabla\bar{\theta}.\nabla\sigma_j.\nabla\theta\mathrm{d}x + \int_{\Omega_j} div(\sigma_j)|\nabla\theta|^2\mathrm{d}x\\ &-\int_{\Gamma_j^0}\sigma_j.\nu_j|\nabla\theta|^2\mathrm{d}\Gamma_j^0 + J_1\\ =&(n-2)\int_{\Omega_j}\psi_j|\nabla\theta|^2\mathrm{d}x\\ &-\int_{\Gamma_j^0}\sigma_j.\nu_j|\nabla\theta|^2\mathrm{d}\Gamma_j^0 + J_1.\end{aligned}$$

Consequently,

$$R_1 = (n-2)\int_{\Omega_j} \psi_j |\nabla\theta|^2 \mathrm{d}x - \int_{\Gamma_j^0} \sigma_j.\nu_j |\nabla\theta|^2 \mathrm{d}\Gamma_j^0 + \Re e(J_1).$$

- **For** R_2

  Integrating I_2 by parts, it follows that:

$$I_2 = -2\mu^2 \int_{\Omega_j} div(\sigma_j\theta)\bar{\theta}\mathrm{d}x + 2\mu^2 \int_{\Gamma_j^0} \sigma_j.\nu_j |\theta|^2 \mathrm{d}\Gamma_j^0$$

Using the fact that

$$div(\sigma_j\theta)\bar{\theta} = (\sigma_j.\nabla\theta)\bar{\theta} + div(\sigma_j)|\theta|^2$$

then, we get

$$\begin{aligned} I_2 =& -2\mu^2 \int_{\Omega_j} (\sigma_j.\nabla\theta)\bar{\theta}\mathrm{d}x - 2\mu^2 \int_{\Omega_j} div(\sigma_j)|\theta|^2 \mathrm{d}x + 2\mu^2 \int_{\Gamma_j^0} \sigma_j.\nu_j |\theta|^2 \mathrm{d}\Gamma_j^0 \\ =& -\overline{I_2} - 2\int_{\Omega_j} div(\sigma_j)|\mu\theta|^2 \mathrm{d}x + 2\int_{\Gamma_j^0} \sigma_j.\nu_j|\mu\theta|^2 \mathrm{d}\Gamma_j^0. \end{aligned}$$

Which implies that

$$R_2 = -n\int_{\Omega_j} \psi_j|\mu\theta|^2 \mathrm{d}x + \int_{\Gamma_j^0} \sigma_j.\nu_j |\mu\theta|^2 \mathrm{d}\Gamma_j^0.$$

Using (3.68), (3.69), R_1 and R_2 in (3.66), we get

$$\begin{aligned} &\Re e(J_1) + (n-2)\int_{\Omega_j} \psi_j|\nabla\theta|^2 \mathrm{d}x - n\int_{\Omega_j} \psi_j|\mu\theta|^2 \mathrm{d}x - \int_{\Gamma_j^0} \sigma_j.\nu_j|\nabla\theta|^2 \mathrm{d}\Gamma_j^0 \\ &+ \int_{\Gamma_j^0} \sigma_j.\nu_j|\mu\theta|^2 \mathrm{d}\Gamma_j^0 = o(1). \end{aligned} \tag{3.70}$$

II. Using the multiplier $2(\sigma_j^T.\nabla_T\bar{\theta})$ on (3.51) over $\Gamma_j^0 \cap \Gamma_1$, it follows that:

$$\begin{aligned} &\Re e\left(2\int_{\Gamma_j^0\cap\Gamma_1} (\sigma_j^T.\nabla_T\bar{\theta})\partial_\nu\theta \mathrm{d}\Gamma_1\right) - \Re e\left(2\int_{\Gamma_j^0\cap\Gamma_1} (\sigma_j^T.\nabla_T\bar{\theta})\Delta_T\theta \mathrm{d}\Gamma_1\right) \\ &- \Re e\left(2\mu^2\int_{\Gamma_j^0\cap\Gamma_1} (\sigma_j^T.\nabla_T\bar{\theta})m(x)\theta \mathrm{d}\Gamma_1\right) = \Re e\left(2\int_{\Gamma_j^0\cap\Gamma_1} (\sigma_j^T.\nabla_T\bar{\theta})G_1 \mathrm{d}\Gamma_1\right) \end{aligned} \tag{3.71}$$

Since σ_j^T is bounded over Γ_1 and, $\nabla_T\theta$ and $\mu\theta$ are uniformly bounded in $L^2(\Gamma_1)$ then considering G_1 in (3.52) with the fact that $m \in L^\infty(\Gamma_1)$ and $\|F\|_{\mathscr{H}} = o(1)$, we deduce that:

$$\Re e\left(2\int_{\Gamma_j^0\cap\Gamma_1} (\sigma_j^T.\nabla_T\bar{\theta})G_1 \mathrm{d}\Gamma_1\right) = o(1). \tag{3.72}$$

Now, set as previously

$$L_1 = 2\int_{\Gamma_j^0\cap\Gamma_1}(\sigma_j^T.\nabla_T\bar{\theta})\partial_\nu\theta d\Gamma_1, \quad L_2 = 2\int_{\Gamma_j^0\cap\Gamma_1}(\sigma_j^T.\nabla_T\bar{\theta})\Delta_T\theta d\Gamma_1,$$

and

$$L_3 = 2\mu^2\int_{\Gamma_j^0\cap\Gamma_1}(\sigma_j^T.\nabla_T\bar{\theta})m(x)\theta d\Gamma_1, \quad R_i^L = \Re e(L_i), i \in \{1,2,3\}.$$

- **For** R_1^L

  Let's skip this case at the moment.

- **For** R_2^L

  Using the integral property of Δ_T given in (3.6), we get

$$L_2 = -2\int_{\Gamma_j^0\cap\Gamma_1}\nabla_T(\sigma_j^T.\nabla_T\bar{\theta}).\nabla_T\theta d\Gamma_1.$$

  As previously, we also have:

$$\nabla_T(\sigma_j^T.\nabla_T\bar{\theta}).\nabla_T\theta = \nabla_T\bar{\theta}.\nabla_T\sigma_j^T.\nabla_T\theta + \frac{1}{2}\sigma_T^T.\nabla_T(|\nabla_T\theta|^2).$$

  Then, we obtain

$$L_2 = (n-3)\int_{\Gamma_j^0\cap\Gamma_1}\psi_j|\nabla_T\theta|^2 d\Gamma_1 + 2\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j B|\nabla_T\theta|^2 d\Gamma_1$$
$$-\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j Tr(B)|\nabla_T\theta|^2 d\Gamma_1.$$

  Finally, $R_2^L = L_2$ since $L_2 \in \mathbb{R}$.

- **For** R_3^L Using an integration by parts, we get

$$L_3 = -2\mu^2\int_{\Gamma_j^0\cap\Gamma_1}\bar{\theta}div_T(\sigma_j^T m(x)\theta)d\Gamma_1.$$

  We have:

$$\bar{\theta}div_T(\sigma_j^T m(x)\theta) = (\sigma_j^T.\nabla_T\theta)m(x)\bar{\theta} + div_T\sigma_j^T m(x)|\theta|^2$$

  Consequently,

$$L_3 = -\overline{L_3} - 2(n-1)\int_{\Gamma_j^0\cap\Gamma_1}\psi_j m(x)|\mu\theta|^2 d\Gamma_1 + 2\int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j Tr(B)m(x)|\mu\theta|^2 d\Gamma_1.$$

  Finally, it follows that

$$R_3^L = -(n-1)\int_{\Gamma_j^0\cap\Gamma_1}\psi_j m(x)|\mu\theta|^2 d\Gamma_1 + \int_{\Gamma_j^0\cap\Gamma_1}\sigma_j.\nu_j Tr(B)m(x)|\mu\theta|^2 d\Gamma_1. \tag{3.73}$$

Replacing R_i^L by their final expressions in (3.71) and thus arranging terms, we obtain:

$$R_1^L + (n-1)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 - (n-3)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j |\nabla_T\theta|^2 \mathrm{d}\Gamma_1 - 2\int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j B|\nabla_T\theta|^2 \mathrm{d}\Gamma_1$$
$$+\int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B)|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 - \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B) m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 = o(1). \tag{3.74}$$

By the computation (3.70) - (3.74), it is obvious that $\Re e(J) - R_1^L$ appears. Which result is given in **Lemma 3.62** via (3.62). Then the subtract (3.70) - (3.74) gives after an arrangement of terms:

$$\begin{aligned}
&(n-2)\int_{\Omega_j} \psi_j|\nabla\theta|^2 \mathrm{d}x - n\int_{\Omega_j} \psi_j|\mu\theta|^2 \mathrm{d}x \\
&+ (n-3)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 - (n-1)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 \\
&- \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j\left[|\nabla\theta|^2 - 2|\partial_{\nu_j}\theta|^2\right] \mathrm{d}\Gamma_j^0 + \int_{\Gamma_j^0} \sigma_j.\nu_j|\mu\theta|^2 \mathrm{d}\Gamma_j^0 \\
&+ 2\int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j B|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 - \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B)|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 \\
&+ \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B) m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 = o(1)
\end{aligned}$$

Multiplying the full estimation by -1 and using the fact that $|\nabla\theta|^2 = |\nabla_T\theta|^2 + |\partial_{\nu_j}\theta|^2$ over $\Gamma_j^0 \cap \Gamma_1$, we get

$$\begin{aligned}
&n\int_{\Omega_j} \psi_j|\mu\theta|^2 \mathrm{d}x - (n-2)\int_{\Omega_j} \psi_j|\nabla\theta|^2 \mathrm{d}x \\
&- (n-3)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 + (n-1)\int_{\Gamma_j^0\cap\Gamma_1} \psi_j m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 \\
&+ \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j|\nabla_T\theta|^2 \mathrm{d}\Gamma_j^0 - \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j|\partial_{\nu_j}\theta|^2 \mathrm{d}\Gamma_j^0 - \int_{\Gamma_j^0} \sigma_j.\nu_j|\mu\theta|^2 \mathrm{d}\Gamma_j^0 \\
&- 2\int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j B|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 + \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B)|\nabla_T\theta|^2 \mathrm{d}\Gamma_1 \\
&- \int_{\Gamma_j^0\cap\Gamma_1} \sigma_j.\nu_j Tr(B) m(x)|\mu\theta|^2 \mathrm{d}\Gamma_1 = o(1)
\end{aligned} \tag{3.75}$$

In order to keep only the first four terms in (3.75), one estimates all the remaining terms. By the definition of ψ_j, it is easily seen that ψ_j over Γ_j^0 is non-vanishing only over $\Gamma_j^c \cap \Gamma$. And since θ vanishes over Γ_0 then the boundary terms in (3.75) are only considered over $\Gamma_j^c \cap \Gamma_1$ where $\sigma_j.\nu_j \leq 0$.

Since σ_j^T, B and $Tr(B)$ are bounded then using **Remark** 3.4.5 and **Remark** 3.4.7, we deduce that

$$\begin{aligned} &n\int_{\Omega_j}\psi_j|\mu\theta|^2\mathrm{d}x-(n-2)\int_{\Omega_j}\psi_j|\nabla\theta|^2\mathrm{d}x\\ &-(n-3)\int_{\Gamma_j^c\cap\Gamma_1}\psi_j|\nabla_T\theta|^2\mathrm{d}\Gamma_1+(n-1)\int_{\Gamma_j^0\cap\Gamma_1}\psi_j m(x)|\mu\theta|^2\mathrm{d}\Gamma_1\\ &-\int_{\Gamma_j^c\cap\Gamma_1}\sigma_j.\nu_j|\partial_{\nu_j}\theta|^2\mathrm{d}\Gamma_j^0=o(1)\end{aligned}$$

Finally, using the fact that $\sigma_j.\nu_j\leq 0$ over Γ_j^c and $\psi_j=1$ over Γ_1, we get the desired result.

- ⋆ **Second step**

The idea is to do the sum while j varies from 1 up to J of (3.65) in order to obtain the same estimation but this time with integrals over Π and Σ. We have

$$\begin{aligned}\Pi&=\Omega\backslash(\Omega\cap\mathcal{O}_0)\\ &=\bigcup_{j=1}^{J}\left(\Omega_j\backslash(\Omega_j\cap\mathcal{O}_0)\right)\end{aligned}$$

As $(\Omega_i\cap\mathcal{O}_0)\cap(\Omega_j\cap\mathcal{O}_0)=\emptyset$ and $\Omega_i\cap\Omega_j=\emptyset$ for all $i\neq j$, then using the definition of ψ_j, we get

$$\begin{aligned}&n\int_{\Pi}|\mu\theta|^2\mathrm{d}\Pi-(n-2)\int_{\Pi}|\nabla\theta|^2\mathrm{d}\Pi+(n-1)\int_{\Sigma}m|\mu\theta|^2\mathrm{d}\Sigma-(n-3)\int_{\Sigma}|\nabla_T\theta|^2\mathrm{d}\Sigma\simeq\\ &n\sum_{j=1}^{J}\int_{\Omega_j}\psi_j|\mu\theta|^2\,\mathrm{d}x-(n-2)\sum_{j=1}^{J}\int_{\Omega_j}\psi_j|\nabla\theta|^2\,\mathrm{d}x+(n-1)\sum_{j=1}^{J}\int_{\Gamma_j^c\cap\Gamma_1}m|\mu\theta|^2\,\mathrm{d}\Gamma_1\\ &-(n-3)\sum_{j=1}^{J}\int_{\Gamma_j^c\cap\Gamma_1}|\nabla_T\theta|^2\,\mathrm{d}\Gamma_1\end{aligned}\tag{3.76}$$

Finally, using (3.65), we get the desired result.

The demonstration of lemma (3.4.9) has been concluded. □

Lemma 3.4.10. *The solution* $(\theta,\eta,\vartheta)^t$ *of the system* (3.1) *satisfies the following estimation*

$$\int_{\Pi}|\mu\theta|^2\mathrm{d}\Pi+\int_{\Pi}|\nabla\theta|^2\mathrm{d}\Pi+\int_{\Sigma}m|\mu\theta|^2\mathrm{d}\Sigma+\int_{\Sigma}|\nabla_T\theta|^2\mathrm{d}\Sigma=o(1).\tag{3.77}$$

where $\Pi=\Omega\backslash(\Omega\cap\mathcal{O}_0)$ *and* $\Sigma=\bigcup_{j=1}^{J}\left(\Gamma_j^c\cap\Gamma_1\right)$.

Proof. Let α be a real number.

Using the multiplier $\alpha\bar{\theta}$ on (3.50), we get

$$\alpha\int_\Omega \bar{\theta}\Delta\theta \mathrm{d}x + \alpha\int_\Omega |\mu\theta|^2\mathrm{d}x - i\alpha\mu\int_\Omega d(x)|\theta|^2\mathrm{d}x = \alpha\int_\Omega \bar{\theta}F_1\mathrm{d}x \tag{3.78}$$

Taking the real part of (3.78) and then using Green's formula with the fact that $\theta_{|\Gamma_0} = 0$ and $\partial_\nu\theta$ given in (3.51), it follows that:

$$\begin{aligned}&-\alpha\int_\Omega |\nabla\theta|^2\mathrm{d}x - \alpha\int_{\Gamma_1} |\nabla_T\theta|^2\mathrm{d}\Gamma_1 + \alpha\int_{\Gamma_1} m(x)|\mu\theta|^2\mathrm{d}\Gamma_1 + \alpha\int_\Omega |\mu\theta|^2\mathrm{d}x\\ &= \alpha\Re e\left(\int_\Omega \bar{\theta}F_1\mathrm{d}x\right) + \alpha\Re e\left(\int_{\Gamma_1} \bar{\theta}G_1\mathrm{d}x\right).\end{aligned}$$

Since $\|F\|_{\mathscr{H}} = o(1)$ and θ uniformly bounded in $L^2(\Omega)$ then

$$-\alpha\int_\Omega |\nabla\theta|^2\mathrm{d}x - \alpha\int_{\Gamma_1} |\nabla_T\theta|^2\mathrm{d}\Gamma_1 + \alpha\int_{\Gamma_1} m(x)|\mu\theta|^2\mathrm{d}\Gamma_1 + \alpha\int_\Omega |\mu\theta|^2\mathrm{d}\Gamma_1 = o(1). \tag{3.79}$$

From (3.53) and (3.57), all terms in (3.79) are $o(1)$ over $\Omega\cap\mathscr{O}_0$. Then, we have

$$\alpha\int_\Pi |\mu\theta|^2\mathrm{d}\Pi - \alpha\int_\Pi |\nabla\theta|^2\mathrm{d}\Pi - \alpha\int_\Sigma |\nabla_T\theta|^2\mathrm{d}\Gamma_1 + \alpha\int_\Sigma m(x)|\mu\theta|^2\mathrm{d}\Sigma = o(1). \tag{3.80}$$

Adding (3.80) and (3.64), we obtain

$$\begin{aligned}&(n+\alpha)\int_\Pi |\mu\theta|^2\mathrm{d}\Pi - (n-2+\alpha)\int_\Pi |\nabla\theta|^2\mathrm{d}\Pi - (n-3+\alpha)\int_\Sigma |\nabla_T\theta|^2\mathrm{d}\Sigma\\ &+ (n-1+\alpha)\int_\Sigma m(x)|\mu\theta|^2\mathrm{d}\Sigma \le o(1)\end{aligned}$$

The choice of $\alpha = \frac{3}{2} - n$ thus provides

$$0 \le \frac{3}{2}\int_\Pi |\mu\theta|^2\mathrm{d}\Pi + \frac{1}{2}\int_\Pi |\nabla\theta|^2\mathrm{d}\Pi + \frac{3}{2}\int_\Sigma |\nabla_T\theta|^2\mathrm{d}\Sigma + \frac{1}{2}\int_\Sigma m(x)|\mu\theta|^2\mathrm{d}\Sigma \le o(1)$$

And this implies (3.77).
The proof is complete. □

Proof. (**Proof of Theorem 3.4.3**)
From (3.53), (3.57), and (3.77) we get $\|\Theta\|_{\mathscr{H}} = o(1) \to 0$ and that is in contradiction with (3.47) (beginning condition on Θ, $\|\Theta\|_{\mathscr{H}} = 1$). So, the condition (3.46) is also satisfied and consequently the energy of the system (3.1) decrease exponentially to 0 while the time t tends to ∞.
The proof is now complete. □

3.5 Observability and controllability

In this section, we first establish the well-posedness of system (3.1) without any control measures. Subsequently, we demonstrate that the control for the initial system (3.1) can be constrained within two bounds related to the initial energy of the homogeneous system, a concept referred to as **Observability results**. These observability outcomes for the homogeneous system enable us to assert the exact controllability of (3.1).

3.5.1 Homogenous system associated to (3.1)

We examine the homogeneous system linked to (3.1) through

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \partial_\nu \varphi + m(x)\varphi_{tt} - \Delta_T \varphi = 0 & \text{on } \mathbb{R}_+ \times \Gamma_1 \\ \varphi = 0 & \text{on } \mathbb{R}_+ \times \Gamma_0 \\ \varphi(0,x) = \varphi_0(x), \ \varphi_t(0,x) = \varphi_1(x) & \text{in } \Omega, \\ \varphi_t(0,x) = \psi_0(x) & \text{in } \Gamma_1. \end{cases} \tag{3.81}$$

Let $(\varphi, \varphi_t, \varphi_{t/\Gamma_1})^t$ be a regular solution of (3.81). The total energy of this system is given by

$$\xi_0(t) = \frac{1}{2}\left[\int_\Omega \left(|\varphi_t|^2 + |\nabla\varphi|^2\right) \mathrm{d}x + \int_{\Gamma_1} \left(|\nabla_T \varphi|^2 + m|\varphi_t|^2\right) \mathrm{d}\Gamma\right]. \tag{3.82}$$

A straightforward computation yields

$$\frac{d}{dt}\xi_0(t) = 0. \tag{3.83}$$

Consequently, the problem (3.81) exhibits conservative characteristics. Reformulating the system (3.81) into Cauchy problem, we derive

$$\begin{cases} \Phi_t = \mathscr{B}\Phi \\ \Phi(0) = \Phi_0 \end{cases} \tag{3.84}$$

where $\Phi = (\varphi, \varphi_t, \varphi_{t/\Gamma_1})^t \in D(\mathscr{B})$ and $\Phi_0 = (\varphi_0, \varphi_1, \psi_0) \in \mathscr{H}$ and $\mathscr{B} : D(\mathscr{B}) \longrightarrow H$ is an operator defined by

$$\mathscr{B} = \begin{pmatrix} 0 & Id & 0 \\ \Delta & 0 & 0 \\ \frac{\Delta_T}{m}{/_{\Gamma_1}} - \frac{\partial_\nu}{m}{/_{\Gamma_1}} & 0 & 0 \end{pmatrix} \tag{3.85}$$

with

$$D(\mathscr{B}) = \left\{\Phi = (\varphi, \zeta, \psi)^t \in \mathscr{H} \;\; \mathscr{B}U \in \mathscr{H};\; \psi = \zeta_{/\Gamma_1},\; (\partial_\nu\varphi + m(x)\varphi_{tt} - \Delta_T\varphi)/_{\Gamma_1} = 0,\; \varphi_{/\Gamma_0} = 0\right\}$$

It is easy to see that for all $\Phi = (\varphi, \varphi_t, \varphi_{t/\Gamma_1})^t \in D(\mathscr{B})$

$$\Re e\langle \mathscr{B}\Phi, \Phi\rangle_{\mathscr{H}} = 0 \leq 0. \tag{3.86}$$

which implies that the operator $\mathscr{B}$ is dissipative. And moreover, we get the existence solution of the following equation thanks to the theorem of **Lax-Milgram**

$$\forall F \in \mathscr{H}, \exists \Phi \in D(\mathscr{B}) \;:\; (I - \mathscr{B})\Phi = F. \tag{3.87}$$

Consequently, the unbounded operator $\mathscr{B}$ is **maximally-monotone** in the space $\mathscr{H}$. Using **Lumer-Philips** theorem, we get the existence and uniqueness solution of (3.84). From (3.83) and (3.15), we finally get the following well-posedness of the homogenous system (3.81).

Theorem 3.5.1.

- *If $\Phi_0 \in \mathscr{H}$ then the system* (3.81) *admits an **unique weak solution*** $\Phi(t)$ *satisfying*

$$\Phi(t) \in C^0((0;T); \mathscr{H}). \tag{3.88}$$

- *If $\Phi_0 \in D(\mathscr{B})$ then the system* (3.81) *admits an **unique strong solution*** $\Phi(t)$ *which satisfy*

$$\Phi(t) \in C^0((0;T); D(\mathscr{B})) \cap C^1((0;T); \mathscr{H}). \tag{3.89}$$

Moreover, we have the following energy conservation property:

$$\|\Phi_0\|^2_{\mathscr{H}} = \|\Phi(t)\|^2_{\mathscr{H}}, \forall t \in \mathbb{R}_+. \tag{3.90}$$

Remark 3.5.2. *Note that $\Phi \in \mathscr{H}$ implies that $\varphi_{t/\Gamma_1} \in L^2(\Gamma_1)$. As a result, $\varphi_{/\Gamma_1} \in H^1(\Gamma_1)$ and for any initial condition $\Phi = (\varphi_0, \varphi_1, \psi_0)^\top \in \mathscr{H}$ such that $\varphi_{0/\Gamma_1} \in H^1(\Gamma_1)$, the solution Φ of the homogenous system satisfies $\varphi_{/\Gamma_1} \in H^1(\Gamma_1)$. Thus, it can be deduced a subtle equivalence as follows:*

$$\Phi = (\varphi, \varphi_t, \varphi_{t/\Gamma_1})^\top \in \mathscr{H} \Longleftrightarrow U = (\varphi, \varphi_t, \varphi_{/\Gamma_1}, \varphi_{t/\Gamma_1})^\top \in \mathbb{H},$$

where $\mathbb{H} = (H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)) \times (H^1(\Gamma_1)) \times L^2(\Gamma_1))$ such that $\|U\|_{\mathbb{H}} = \|\Phi\|_{\mathscr{H}}$.

Moreover, $\varphi_{/\Gamma_1} \in H^2(\Gamma_1)$ for strong solutions Φ since φ_{tt/Γ_1} is then in $L^2(\Gamma_1)$.

3.5.2 Observability results

In this part, we will establish an indirect and direct observability inequality. The indirect observability inequality will be used to show the existence and uniqueness solution of reverse time problem which will allow us to have the exact controllability of (3.1). For this aim, we use the study done by **A. Haraux** in [41] especially the **Proposition 2** and the **Theorem 4.1** in [77] for which the exponential stability of (3.1) implies the existence of two positive constants a, b ($a \leq b$) such that for initial condition $\Phi_0 = (\varphi_0, \varphi_1, \psi_0)^\top \in \mathscr{H}$, the solution $\Phi(t,x) = (\varphi, \varphi_t, \varphi_{t/_{\Gamma_1}})^\top$ of (3.81) satisfies

$$a\,\|\Phi_0\|^2_{\mathscr{H}} \leq \int_0^T \int_\Omega d(x)|\varphi_t|^2 \mathrm{d}x\mathrm{d}t \leq b\,\|\Phi_0\|^2_{\mathscr{H}}, \quad T > T_0 \text{ (large enough)}. \tag{3.91}$$

3.5.3 Exact controllability

In this section, we will study the exact controllability of the wave equation influenced by an internal dynamic controller using HUM method [50, 65]. To do this, we consider the following system

$$\begin{cases} \theta_{tt} - \Delta\theta = dv & in\ \mathbb{R}_+ \times \Omega \\ \partial_\nu\theta + m(x)\theta_{tt} - \Delta_T\theta = 0 & on\ \mathbb{R}_+ \times \Gamma_1 \\ \theta = 0 & on\ \mathbb{R}_+ \times \Gamma_0 \\ \theta(0,x) = \theta_0(x),\ \theta_t(0,x) = \theta_1(x) & in\ \Omega \\ \theta_t(0,x) = \vartheta_0(x) & on\ \Gamma_1 \end{cases} \tag{3.92}$$

Here, we aim to ascertain whether, for $T \geq T_0 > 0$ sufficiently large and all initial conditions $\Theta_0 = (\theta_0, \theta_1, \vartheta_0)^\top$, there exists a control v capable of steering the solution of the system denoted by (3.92) back to equilibrium at time T (finite). In other words, we investigate whether the solution of the problem expressed in (3.1) satisfies $\theta(T) = \theta_t(T) = \theta_t(T)_{/_{\Gamma_1}} = 0$. For this aim, we start to show the existence and uniqueness weak solution of the system (3.92). And for this, the following result is required.

Lemma 3.5.3. *Let $T > 0$ and $\Phi = (\varphi, \varphi_t, \varphi_{t/_{\Gamma_1}})^\top \in \mathscr{H}$ be the solution of the homogenous system* (3.81) *associated to the initial condition $\Phi_0 \in \mathscr{H}$. The following dual result thus hold for $\theta = \theta(t,x)$ of* *system* (3.92)*:*

$$\left\langle \begin{pmatrix} \theta_t(T) \\ -\theta(T) \\ \theta_t(T)_{/\Gamma_1} \\ -\theta(T)_{/\Gamma_1} \end{pmatrix}, \begin{pmatrix} \varphi(T) \\ \varphi_t(T) \\ \varphi(T)_{/\Gamma_1} \\ \varphi_t(T)_{/\Gamma_1} \end{pmatrix} \right\rangle_{\mathbb{H}'\times\mathbb{H}} = \left\langle \begin{pmatrix} \theta_1 \\ -\theta_0 \\ \vartheta_0 \\ -\theta_{0/\Gamma_1} \end{pmatrix}, \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_{0/\Gamma_1} \\ \psi_0 \end{pmatrix} \right\rangle_{\mathbb{H}'\times\mathbb{H}} + \int_0^T \int_\Omega d(x)v(t)\varphi(t)\mathrm{d}x\mathrm{d}t \quad (3.93)$$

where $\mathbb{H} = (H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)) \times (H^1(\Gamma_1) \times L^2(\Gamma_1))$ *and* $\mathbb{H}'$ *its dual,*
and $\langle .,. \rangle_{\mathbb{H}'\times\mathbb{H}}$ *denotes the inner scalar product of* $\mathbb{H}' \times \mathbb{H}$ *defined by:*

$$\langle U, V \rangle_{\mathbb{H}'\times\mathbb{H}} = \langle u_1, v_1 \rangle_{L^2(\Omega)} + \langle u_2, v_2 \rangle_{L^2(\Omega)} + \langle m(x)u_3, v_3 \rangle_{L^2(\Gamma_1)} + \langle m(x)u_4, v_4 \rangle_{L^2(\Gamma_1)}$$

for all $U = (u_1, u_2, u_3, u_4)^\top \in \mathbb{H}'$ *and* $V = (v_1, v_2, v_3, v_4)^\top \in \mathbb{H}$.

Proof. Set $\Omega_T = \Omega \times (0,T)$ and $\Gamma_{1T} = \Gamma_1 \times (0,T)$, for $T > 0$.

Multiplying $(3.92)_1$ by φ and then integrating over Ω_T, we get

$$I_1 + I_2 = \int_{\Omega_T} d(x)v\varphi \mathrm{d}x\mathrm{d}t. \quad (3.94)$$

where $I_1 = \int_{\Omega_T} \theta_{tt}\varphi \mathrm{d}x\mathrm{d}t$ and $I_2 = -\int_{\Omega_T} \varphi\Delta\theta \mathrm{d}x\mathrm{d}t$.

- **For** I_1

  Using a double integration in t by parts, it follows that

$$\begin{aligned} I_1 &= \left[\int_\Omega \theta_t(t)\varphi(t)\mathrm{d}x\right]_0^T - \int_{\Omega_T} \theta_t\varphi_t \mathrm{d}x\mathrm{d}t \\ &= \left[\int_\Omega \theta_t(t)\varphi(t)\mathrm{d}x\right]_0^T - \left[\int_\Omega \theta(t)\varphi_t(t)\mathrm{d}x\right]_0^T + \int_{\Omega_T} \theta\varphi_{tt}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

  Then

$$I_1 = \left[\int_\Omega [\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T + \int_{\Omega_T} \theta\varphi_{tt}\mathrm{d}x\mathrm{d}t \quad (3.95)$$

- **For** I_2

  Using the formula of Green and the fact that $\theta_{/\Gamma_0} = 0$, it follows that

$$I_2 = \int_{\Omega_T} \nabla\theta.\nabla\varphi \mathrm{d}x\mathrm{d}t - \int_{\Gamma_{1T}} \varphi\partial_\nu\theta \mathrm{d}\Gamma_1\mathrm{d}t$$

  Integrating by parts over Ω, we get

$$\begin{aligned} I_2 &= -\int_{\Omega_T} \theta div(\nabla\varphi)\mathrm{d}x\mathrm{d}t + \int_{\Gamma_{1T}} \theta\nabla\varphi.\nu \mathrm{d}\Gamma_1\mathrm{d}t - \int_{\Gamma_{1T}} \varphi\partial_\nu\theta \mathrm{d}\Gamma_1\mathrm{d}t \\ &= -\int_{\Omega_T} \theta\Delta\varphi \mathrm{d}x\mathrm{d}t + \int_{\Gamma_{1T}} \theta\partial_\nu\varphi \mathrm{d}\Gamma_1\mathrm{d}t - \int_{\Gamma_{1T}} \varphi\partial_\nu\theta \mathrm{d}\Gamma_1\mathrm{d}t \end{aligned}$$

Considering both $\partial_\nu\theta$ and $\partial_\nu\varphi$ over Γ_1 and the integral property of Δ_T given in (3.6), it follows that

$$\begin{aligned} I_2 &= -\int_{\Omega_T} \theta\Delta\varphi \mathrm{d}x\mathrm{d}t + \int_{\Gamma_{1T}} \theta[\Delta_T\varphi - m(x)\varphi_{tt}]\mathrm{d}\Gamma_1\mathrm{d}t - \int_{\Gamma_{1T}} \varphi[\Delta_T\theta - m(x)\theta_{tt}]\mathrm{d}\Gamma_1\mathrm{d}t \\ &= -\int_{\Omega_T} \theta\Delta\varphi \mathrm{d}x\mathrm{d}t + \underbrace{\int_{\Gamma_{1T}} m(x)\theta_{tt}\varphi\mathrm{d}\Gamma_1\mathrm{d}t - \int_{\Gamma_{1T}} m(x)\varphi_{tt}\theta\mathrm{d}\Gamma_1\mathrm{d}t}_{:=J} \end{aligned}$$

Using a double integration in t by parts as previously in I_1, we deduce that

$$J = \left[\int_{\Gamma_1} m(x)[\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T + \int_{\Omega_T} m(x)\theta\varphi_{tt}\mathrm{d}x\mathrm{d}t$$

As a result, it follows that

$$I_2 = -\int_{\Omega_T} \theta\Delta\varphi\mathrm{d}x\mathrm{d}t + \left[\int_{\Gamma_1} m(x)[\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T \tag{3.96}$$

Now using I_1 and I_2 in (3.94), we get

$$\begin{aligned} &\left[\int_{\Omega} \theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)\mathrm{d}x\right]_0^T + \left[\int_{\Gamma_1} m(x)[\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T \\ &+ \int_{\Omega_T} \theta(\varphi_{tt} - \Delta\varphi)\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} d(x)v\varphi\mathrm{d}x\mathrm{d}t \end{aligned}$$

Since $\Phi = (\varphi, \varphi_t, \varphi_{t/\Gamma_1})$ is the homogenous system's solution, then $\varphi_{tt} - \Delta\varphi = 0$ over Ω_T. As a result,

$$\left[\int_{\Omega} [\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T + \left[\int_{\Gamma_1} m(x)[\theta_t(t)\varphi(t) - \theta(t)\varphi_t(t)]\mathrm{d}x\right]_0^T = \int_{\Omega_T} d(x)v\varphi\mathrm{d}x\mathrm{d}t$$

which is

$$\begin{aligned} \int_{\Omega} [\theta_t(T)\varphi(T) - \theta(T)\varphi_t(T)]\mathrm{d}x + \int_{\Gamma_1} m(x)[\theta_t(T)\varphi(T) - \theta(T)\varphi_t(T)]\mathrm{d}x = &\int_{\Omega} [\theta_t(0)\varphi(0) - \theta(0)\varphi_t(0)]\mathrm{d}x \\ &+ \int_{\Gamma_1} m(x)[\theta_t(0)\varphi(0) - \theta(0)\varphi_t(0)]\mathrm{d}x \\ &+ \int_{\Omega_T} d(x)v\varphi\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The proof is complete. □

Now, we are in position to state the well-posedness of system (3.92).

Theorem 3.5.4. *Let $T \geq T_0 > 0$ large enough. Assume that ω satisfies the geometric condition* ***PMGC*** *and let $\Theta_0 = (\theta_1, \theta_0, \vartheta_0, \theta_{0/\Gamma_1})^\top \in \mathbb{H}'$, $v(t) = -\frac{d}{dt}v_0(t) \in H^1(0;T;L^2(\omega))'$. The controlled system* (3.92) *has an unique weak solution*

$$\Theta = (\theta_t, -\theta, \theta_{t/\Gamma_1}, -\theta_{/\Gamma_1})^\top \in C^0([0;T];\mathbb{H}'). \tag{3.97}$$

Proof. Define the linear map $\mathbb{L}$ by

$$\mathbb{L}(\Phi_0) = \langle \Theta(0), \Phi_0 \rangle_{\mathbb{H}' \times \mathbb{H}} + \int_0^T \int_\Omega d v(t)\varphi(t) \mathrm{d}x\mathrm{d}t, \forall \Phi_0 \in \mathbb{H}. \tag{3.98}$$

Then from (3.93), we obtain the following variational problem

$$\mathbb{L}(\Phi_0) = \langle \Theta(T), S_{\mathscr{B}}(T)\Phi_0 \rangle_{\mathbb{H}' \times \mathbb{H}}, \forall \Phi_0 \in \mathbb{H}. \tag{3.99}$$

where $(S_{\mathscr{B}}(t))_{t \geq 0}$ is the C_0 semi-group of contraction generates by $\mathscr{B}$.
Let $v_0 \in L^2(0;T;L^2(\omega))$, we choose the control $v(t)$ as follows

$$v(t) = -\frac{d}{dt}v_0(t) \in (H^1(0;T;L^2(\omega)))'. \tag{3.100}$$

where the derivative with respect to t is taken in the sense of the duality $H^1(0;T;L^2(\omega))$ and its dual $(H^1(0;T;L^2(\omega)))'$, that is

$$-\int_0^T \frac{d}{dt}v_0(t)u(t)\mathrm{d}t = \int_0^T v_0(t)\frac{d}{dt}u(t)\mathrm{d}t, \forall u \in (H^1(0;T;L^2(\omega)))'. \tag{3.101}$$

Using the Cauchy-Schwarz inequality and the definition of d, we get

$$\begin{aligned}\left|\int_0^T \int_\Omega d(x)v(t)\varphi(t)\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^T \int_\Omega d(x)v_0(t)\varphi_t(t)\mathrm{d}x\mathrm{d}t\right| \\ &\leq \|v_0\|_{L^2(0;T;L^2(\omega))} \left\|d^{\frac{1}{2}}\varphi_t\right\|_{L^2(0;T;L^2(\Omega))}\end{aligned}$$

Using the inequality of observability given in (3.91) and recalling **Remark** 3.5.2, we get

$$\left|\int_0^T \int_\Omega d(x)v(t)\varphi(t)\mathrm{d}x\mathrm{d}t\right| \leq b\,\|v_0\|_{L^2(0;T;L^2(\omega))}\,\|\Phi_0\|_{\mathbb{H}}. \tag{3.102}$$

Using (3.102) in (3.98), we get

$$|\mathbb{L}(\Phi_0)| \leq \left(b\,\|v_0\|_{L^2(0;T;L^2(\omega))} + \|\Theta_0\|_{\mathbb{H}'}\right)\|\Phi_0\|_{\mathbb{H}}, \forall \Phi_0 \in \mathbb{H}. \tag{3.103}$$

Which implies the continuity of the linear map $\mathbb{L}$ in $\mathbb{H}$. Moreover, we have

$$\|\Theta(T)\|_{\mathbb{H}'} = \|\mathbb{L}\|_{\mathscr{L}(\mathbb{H},\mathbb{R})} \leq b\,\|v_0\|_{L^2(0;T;L^2(\omega))} + \|\Theta_0\|_{\mathbb{H}'} \tag{3.104}$$

Using Riesz representation theorem, there exists a unique $\mathbb{Z}(t,x) \in \mathbb{H}'$ such that

$$\mathbb{L}(\Phi_0) = \langle \mathbb{Z}, \Phi_0 \rangle_{\mathbb{H}' \times \mathbb{H}}, \quad \forall \Phi_0 \in \mathbb{H}.$$

Then using (3.99), it follows that (3.92) admits a unique weak solution $\Theta(t,x) = \mathbb{Z}(t,x)$. The proof is complete. □

Theorem 3.5.5. *Assume that ω satisfies the geometric condition (**PMGC**). For all $T \geq T_0 > 0$, T_0 large enough and for all $\Theta_0 \in \mathbb{H}'$ there exists a control $v(t) \in (H^1(0;T;L^2(\omega)))'$ such that the solution Θ of the controlled system* (3.92) *satisfies $\theta(T) = \theta_t(T) = \theta(T)_{/\Gamma_1} = \theta_t(T)_{/\Gamma_1} = 0$.*

Proof. The observability inequalities in (3.91) in $\mathbb{H}$ provide a result of equivalent norms, more precisely between $\|.\|_{\mathbb{H}}$ and $\|.\|_{L^2(\omega)}$. This allows us to consider $\|.\|_{\mathbb{H}}$ as the semi-norm

$$\|\Phi_0\|_{\mathbb{H}} = \left(\int_0^T \int_\omega |\varphi_t|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}, \tag{3.105}$$

where $\Phi = (\varphi, \varphi_t, \varphi_{/\Gamma_1}, \varphi_{t/\Gamma_1})^\top \in \mathbb{H}$ is the solution of the homogenous system (3.81) associated to the initial condition $\Phi_0 \in \mathbb{H}$. We choose the control $v = \frac{d}{dt}\varphi_t \in (H^1(0;T;L^2(\omega)))'$ and then solve the time reverse problem

$$\begin{cases} \varsigma_{tt} - \Delta\varsigma = d\dfrac{d}{dt}\varphi_t & \textit{in } \mathbb{R}_+ \times \Omega \\ \partial_\nu \varsigma + m(x)\varsigma_{tt} - \Delta_T \varsigma = 0 & \textit{on } \mathbb{R}_+ \times \Gamma_1 \\ \partial_\nu \varsigma = 0 & \textit{on } \mathbb{R}_+ \times \Gamma_0 \\ \varsigma(T,x) = 0, \ \varsigma_t(T,x) = 0 & \textit{in } \Omega, \\ \varsigma_t(T,x) = 0, & \textit{on } \Gamma_1. \end{cases} \tag{3.106}$$

The theorem 3.5.4 provides the existence and uniqueness of system (3.92)'s solution $\Xi = \Xi(t)$ such that

$$\Xi = (\varsigma_t, -\varsigma, \varsigma_{t/\Gamma_1}, -\varsigma_{/\Gamma_1})^\top \in C^0([0;T];\mathbb{H}').$$

Moreover, we have

$$\|\Xi\|_{\mathbb{H}'} \leq b\,\|\varphi_t\|_{L^2(0;T;L^2(\omega))} + \|\Xi(0)\|_{\mathbb{H}'} \tag{3.107}$$

We define the linear operator $\mathscr{T} : \mathbb{H} \longrightarrow \mathbb{H}'$ such that

$$\mathscr{T}\Phi_0 = \Xi(0) = (\varsigma_1, -\varsigma_0, \varsigma_{1/\Gamma_1}, -\varsigma_{0/\Gamma_1})^\top \in \mathbb{H}', \quad \forall \Phi_0 \in \mathbb{H}.$$

Now, consider the following variational problem associated to the semi-norm given in (3.105)

$$\left\langle \mathscr{T}\Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbb{H}'\times\mathbb{H}} = \int_0^T \int_\omega \varphi_t \tilde{\varphi}_t \mathrm{d}x\mathrm{d}t = \left\langle \Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbb{H}}, \forall \widetilde{\Phi}_0 \in \mathbb{H}, \tag{3.108}$$

Using the inequality of Cauchy-Schwarz on (3.108), we deduce that

$$\left| \left\langle \mathscr{T}\Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbb{H}'\times\mathbb{H}} \right| \le \|\Phi_0\|_{\mathbb{H}} \left\| \widetilde{\Phi}_0 \right\|_{\mathbb{H}}, \forall \Phi_0, \widetilde{\Phi}_0 \in \mathbb{H}. \tag{3.109}$$

In particular, we have

$$\left| \langle \mathscr{T}\Phi_0, \Phi_0 \rangle_{\mathbb{H}'\times\mathbb{H}} \right| = \|\Phi_0\|^2_{\mathbb{H}}, \forall \Phi_0 \in \mathbb{H}. \tag{3.110}$$

Then the bi-linear map

$$(\Phi_0, \widetilde{\Phi_0}) \longrightarrow \left\langle \mathscr{T}\Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbb{H}'\times\mathbb{H}} \tag{3.111}$$

is continuous and coercive on $\mathbb{H}' \times \mathbb{H}$. Then according to the lemma of **Lax-Milgram**, $\mathscr{T}$ is an isomorphism from $\mathbb{H}$ on $\mathbb{H}'$. In particular, for each $\Theta_0 = (\theta_1, -\theta_0, \theta_{1/\Gamma_1}, -\theta_{0/\Gamma_1})^\top \in \mathbb{H}'$, there exists a unique $\Phi_0 \in \mathbb{H}$ such that

$$\mathscr{T}\Phi_0 = \Theta_0.$$

Since for all Φ_0, $\mathscr{T}\Phi_0 = \Xi_0$, then $\Xi_0 = \Theta_0$. Consequently, the uniqueness result of system (3.92)'s solution for every $\Theta_0 \in \mathbb{H}'$ thus gives us

$$\Theta = \Xi, \tag{3.112}$$

which means

$$\Theta(T) = 0 \Longleftrightarrow \theta(T) = \theta(T)_{/\Gamma_1} = \theta_t(T) = \theta_t(T)_{/\Gamma_1} = 0. \tag{3.113}$$

The proof is complete. □

3.6 Conclusion

In conclusion, owing to the well-posedness of system (3.1) and the strong stability result, the construction of exponential stability has been achieved. The choice of a sufficiently large damped region ω along Γ_1 was made so as to satisfy the PMGC condition. This made it possible to establish the exponential stability of the system, which in turn allows one to deduce its observability and, subsequently, its controllability.

4

Stabilization of a Locally Damped Wave with Negligible Inertia at the Boundary: Method of Nakao

This chapter studies the exponential stability of an internal locally damped wave, by the use of Nakao's method. The medium Ω is a thin annular corpus with negligible mass density over its outer boundary Γ_1, that is, Γ_1 tends to follow the outcoming waves dynamic.

Before starting, let's define the considered shape of Ω and that of the damped region ω.

4.1 Geometry of the medium

Let $B_s(a)$ denote the open ball with center $a \in \mathbb{R}^n$ and radius $s > 0$ in the Euclidean space $(\mathbb{R}^n, \|.\|_2)$, where $n \geq 2$. We define $\Omega \subset \mathbb{R}^n$ as the n-dimensional annular domain $B_1(0)\backslash\overline{B_{r_0}(0)}$, with $0 < r_0 < 1$, bounded by two smooth boundaries:

$$\Gamma_0 = \partial B_{r_0}(0) \quad \text{and} \quad \Gamma_1 = \partial B_1(0).$$

Let $\omega \subset \Omega$ be the n-dimensional sub-annular domain $B_1(0)\backslash\overline{B_r(0)}$, where $r_0 < r < 1$. It is bounded by Γ_1 and the boundary $\mathscr{I} = \partial B_r(0)$, is contained within Ω.

Then, ω and associated geometric domains are defined as follows:

$$\Omega = \{x \in \mathbb{R}^n \,:\, 0 < r_0 < \|x\|_2 < 1\} \subset \mathbb{R}^n, \quad \omega = \{x \in \mathbb{R}^n \,:\, r < \|x\|_2 < 1, r_0 < r < 1\},$$

and

$$\Gamma_0 = \{x \in \mathbb{R}^n \,:\, \|x\|_2 = r_0 > 0\}$$
$$\Gamma_1 = \{x \in \mathbb{R}^n \,:\, \|x\|_2 = 1\}$$
$$\mathscr{I} = \{x \in \mathbb{R}^n \,:\, \|x\|_2 = r, 0 < r_0 < r < 1\}$$

satisfying the conditions:

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$$
$$\overline{\Gamma_i} \cap \overline{\mathscr{I}} = \emptyset, \forall i \in \{0, 1\}$$
$$\partial\omega = \Gamma_1 \cup \mathscr{I} \,(\omega \subset \Omega \text{ is strict}).$$

This chapter is merely focused on the published paper [13].

4.2 MODEL (4.1)

We first consider a free propagation of the wave $\vartheta = \vartheta(x,t)$ in Ω until it reaches a certain neighborhood of Γ_1 in Ω. In this neighborhood $\omega \subset \Omega$, the wave is weakened by a dissipation mechanism related to the wave's speed $d\vartheta_t$ where $d = d_0\chi_\omega$, $d_0 > 0$. In addition, the wave is under a certain geometric condition over the border Γ_1. Which is, the changes in wave's displacement toward outward normal direction ($\partial_\nu\vartheta$) is expressed with the changes of tangential slope along Γ_1 ($\Delta_T\vartheta = div_T(\nabla_T\vartheta)$). This allows a certain flexibility of Γ_1 that isn't present in the case where $\vartheta_{|\Gamma_1}$ is constant.

Here is the system thus obtained by such a disturbance in $\overline{\Omega}$ over time:

$$\begin{cases} \vartheta_{tt}(x,t) - \Delta\vartheta(x,t) + d(x)\vartheta_t(x,t) = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \vartheta = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}_+, \\ \partial_\nu\vartheta = \Delta_T\vartheta(x,t), & (x,t) \in \Gamma_1 \times \mathbb{R}_+, \\ \vartheta(x,0) = \vartheta_0(x), \quad \vartheta_t(x,0) = \vartheta_1(x), & x \in \overline{\Omega}, \end{cases} \tag{4.1}$$

where $\vartheta_0 = \vartheta_0(x)$ is the initial position of the wave ϑ,
$\vartheta_1 = \vartheta_1(x)$ initial wave's speed, and the couple $(\vartheta_0, \vartheta_1)^\top$ is the initial state of the system (4.1). The operator of Laplace-Beltrami Δ_T [90] satisfies the following property:

$$\int_{\Gamma_1} v\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} \nabla_T u \nabla_T v \mathrm{d}\Gamma_1, \quad \forall u, v \in H^1(\Gamma_1), \tag{4.2}$$

where ∇_T represents the tangential gradient operator over Γ_1.

In particular,

$$\int_{\Gamma_1} u\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} |\nabla_T u|^2 \mathrm{d}\Gamma_1, \quad \forall u \in H^1(\Gamma_1). \tag{4.3}$$

Next, let's consider the minimal state space of (4.1) as $\mathscr{H} = \mathscr{V} \times L^2(\Omega)$ where

$$\mathscr{V} = \left\{ z \in H^1_{\Gamma_0}(\Omega) \ : \ z_{/\Gamma_1} \in H^1(\Gamma_1) \right\}.$$

The associated norms are defined as

$$\|\vartheta\|^2_{\mathscr{V}} = \|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)}, \quad \|(\vartheta,\, v)\|^2_{\mathscr{H}} = \|\vartheta\|^2_{\mathscr{V}} + \|v\|^2_{L^2(\Omega)},$$

for all $(\vartheta, v) \in \mathscr{V} \times L^2(\Omega)$.

Remark 4.2.1. *As previously in **Chapter** 2, it follows from (2.3.14) that $\mathscr{V} \subset H^1_{\Gamma_0}(\Omega)$. By taking $\|.\|_{\mathscr{V}}$ as an equivalent norm to that of $\|.\|_{H^1_{\Gamma_0}(\Omega)}$, poincaré's inequality thus remains valid for any element of $\mathscr{V}$.*

Now, we are in position to establish the well-posedness of the system (4.1).

4.3 WELL-POSEDNESS OF THE SYSTEM

In this section, we explore the existence and uniqueness of solutions for the system (4.1), followed by an examination of its stability. For this analysis, we use the semi-group method.

Assume that $U_0 = (\vartheta_0, \vartheta_1)^\top \in \mathscr{H}$. The system (4.1) can be reformulate into a first order Cauchy problem. For this aim, let's define $U = (\vartheta, v)^\top \in \mathscr{H}$ with $v = \vartheta_t$.
Then, $U_t = (v, v_t = \vartheta_{tt})^\top \in \mathscr{H}$.
From first equation of (4.1), it follows that:

$$U_t = \underbrace{(v, \Delta\vartheta - d(x)v)^\top}_{:=AU}.$$

Consequently, the system (4.1) is rewritten as:

$$\begin{cases} U_t = AU \\ U(0) = U_0 \end{cases} \tag{4.4}$$

where the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$A = \begin{pmatrix} 0 & I \\ \Delta & -d(x)I \end{pmatrix} \tag{4.5}$$

where $D(A) = \{U = (\vartheta, v) \in \mathcal{H} \ : \ v \in \mathcal{V}; \quad \Delta\vartheta - d(x)v \in L^2(\Omega)\}$.

Here, I denotes the identity operator.

Lemma 4.3.1. *(dissipation of A)*

The linear operator A from the problem (4.4) *is dissipative, meaning that:*

$$\Re e\langle AU, U\rangle_{\mathcal{H}} \leq 0, \quad \forall U \in \mathcal{H}. \tag{4.6}$$

where $\langle ., .\rangle_{\mathcal{H}}$ *is the associated scalar product with the norm* $\|.\|_{\mathcal{H}}$*.*

Proof. Let $U = (\vartheta, v)^\top \in \mathcal{H}$. By Green's formula, property (4.2) and using the boundary conditions, we see that:

$$\begin{aligned} \Re e\langle AU, U\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ \Delta\vartheta - d(x)v \end{pmatrix}, \begin{pmatrix} \vartheta \\ v \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \int_\Omega \nabla v \nabla\vartheta \, \mathrm{d}x + \int_{\Gamma_1} \nabla_T v \nabla_T \vartheta \, \mathrm{d}\Gamma_1 \\ &+ \int_\Omega v(\Delta\vartheta - d(x)v) \, \mathrm{d}x \\ &= -\int_\Omega d(x)|v|^2 \mathrm{d}x \end{aligned} \tag{4.7}$$

Since $d(x) \geq 0$ over Ω, we get (4.6).

The proof is achieved. □

Theorem 4.3.2. *(Existence and uniquess)*

The system (4.1) *has a unique weak solution* $U \in C([0, +\infty[, \mathcal{H})$ *for* $U_0 \in \mathcal{H}$*.*

In addition, if $U_0 \in D(A)$*, then the system* (4.1) *admit a unique strong solution U such that*

$$U \in C^0([0, +\infty[, D(A)) \cap C^1([0, +\infty[, \mathcal{H}).$$

Proof. According to **lemma** 4.6, the linear operator A is dissipative. Additionally, if for all $\lambda > 0$, the operator $\lambda I - A$ is surjective, then Lumer-Philips theorem guarantees the existence and uniqueness solution to the associated evolution problem.

To prove this, we need to show that for all $F = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{H}$ there exists $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ such that:

$$(\lambda I - A)U = F. \tag{4.8}$$

That is,

$$\begin{cases} \lambda u - v = f \\ (\lambda + d(x))v - \Delta u = g \end{cases} \tag{4.9}$$

From the 1^{st} equation of the system (4.9), we get:

$$v = \lambda u - f \tag{4.10}$$

Combining (4.10) and the 2^{nd} equation of (4.9), we get:

$$a(x)u - \Delta u = h \tag{4.11}$$

where $a(x) = \lambda(\lambda + d(x)) > 0$ and $h = g + (\lambda + d(x))f$.
Multiplying (4.11) by $z \in \mathscr{V}$ and then integrating by parts over Ω, we obtain:

$$\varphi(u, z) = L(z) \tag{4.12}$$

where the bilinear form $\varphi(.,.)$ and the linear form $L(.)$ are given as follows

$$\begin{cases} \varphi(u, z) = \displaystyle\int_\Omega a(x)uz \,\mathrm{d}x + \int_\Omega \nabla u \nabla z \,\mathrm{d}x + \int_{\Gamma_1} \nabla_T u \nabla_T z \,\mathrm{d}\Gamma_1 \\ L(z) = \displaystyle\int_\Omega hz \,\mathrm{d}x \end{cases} \tag{4.13}$$

Since $\varphi(.,.)$ is clearly coercive over $\mathscr{V}$:

$$|\varphi(u, u)| = \int_\Omega a(x)|u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\Gamma_1} |\nabla_T u|^2 \,\mathrm{d}\Gamma_1 \geq 1.\|u\|^2_{\mathscr{V}}.$$

and L is continuous on $\mathscr{V}$, the theorem of Lax-Milgram confirm that the problem (4.12) has a unique solution $u \in \mathscr{V}$. In order to prove that $U = (u, v)^\top \in D(A)$, we need to show that $v \in \mathscr{V}$ and $\Delta u - d(x)v \in L^2(\Omega)$.

- Since $H^1(\Omega)$ and $H^1(\Gamma_1)$ are vector spaces and $v = \lambda u - f$ then using the fact that $u, f \in \mathscr{V}$, we get $(v, v_{/\Gamma_1}) \in H^1(\Omega) \times H^1(\Gamma_1)$ and $v_{/\Gamma_0} = \lambda u_{/\Gamma_0} - f_{/\Gamma_0} = 0$.
  Hence $v \in \mathscr{V}$.

- Since $v \in \mathscr{V} \subset L^2(\Omega)$ and $g \in L^2(\Omega)$, we conclude that $\Delta u - d(x)v \in L^2(\Omega)$. Moreover, the condition $\partial_\nu u = \Delta_T u$ is natural for any strong solution u (see [17]).

Finally, $U = (u, v)^\top \in D(A)$ which satisfies (4.9). This shows that the operator A is maximal-dissipative on $\mathscr{H}$ and then generates a C_0-semigroup of contraction $(S(t))_{t\geq 0}$ in $\mathscr{H}$. Thanks to the theorem of Lumer-Philips, the problem (4.1) admits a unique solution expressed as $U(t) = S(t)U_0$. □

Now, let's study the stability of this solution.

Proposition 4.3.3. *(Decay result)*
Let $(\vartheta, \vartheta_t)^\top \in \mathscr{H}$ *be a solution of the system* (4.1). *The released energy* ξ *defined as:*

$$\xi(t) = \frac{1}{2}\left\|(\vartheta, \vartheta_t)^\top\right\|_{\mathscr{H}}^2, \tag{4.14}$$

decreases over time such that:

$$\frac{d\xi(t)}{dt} = -\int_\Omega d(x)|\vartheta_t|^2 \mathrm{d}x. \tag{4.15}$$

Proof. Multiplying the first equation of (4.1) by ϑ_t and integrating over Ω, we obtain

$$\int_\Omega \vartheta_t \vartheta_{tt}\,\mathrm{d}x - \int_\Omega \vartheta_t \Delta\vartheta\,\mathrm{d}x + \int_\Omega d(x)|\vartheta_t|^2\,\mathrm{d}x = 0 \tag{4.16}$$

1. **First term in** (4.16)**:**

   Using the identity $\frac{d}{dt}(|\vartheta_t|^2) = 2\vartheta_t\vartheta_{tt}$, we get

$$\int_\Omega \vartheta_t\vartheta_{tt}\,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\vartheta_t|^2\,\mathrm{d}x\right). \tag{4.17}$$

2. **Second term in** (4.16)**:**

   Using Green's formula, and the boundary conditions, we get:

$$-\int_\Omega \vartheta_t\Delta\vartheta\,\mathrm{d}x = \int_\Omega \nabla\vartheta\nabla\vartheta_t\,\mathrm{d}x + \int_{\Gamma_1} \nabla_T\vartheta\nabla_T\vartheta_t\,\mathrm{d}\Gamma_1.$$

   Using the product rule for the time derivative of the gradient norms, we get:

$$\int_\Omega \nabla\vartheta\nabla\vartheta_t\,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x\right), \quad \int_{\Gamma_1} \nabla_T\vartheta\nabla_T\vartheta_t\,\mathrm{d}\Gamma_1 = \frac{1}{2}\frac{d}{dt}\left(\int_{\Gamma_1} |\nabla_T\vartheta|^2\,\mathrm{d}\Gamma_1\right).$$

   Combining all the term in (4.16), we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\vartheta_t|^2\,\mathrm{d}x + \int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x + \int_{\Gamma_1} |\nabla_T\vartheta|^2\,\mathrm{d}\Gamma_1\right) = -\int_\Omega d(x)|\vartheta_t|^2\,\mathrm{d}x, \tag{4.18}$$

which is:

$$\frac{d}{dt}\xi(t) = -\int_{\Omega} d(x)|\vartheta_t|^2 \, \mathrm{d}x \le 0.$$

This implies ξ decreases over time, since $\xi'(t) = \frac{d}{dt}\xi(t) \le 0$.

The proof is complete. □

Finally, the system (4.1) is well-posed in $\mathscr{H}$.

4.4 Exponential Stability - Nakao's Method

In this section, we establish an exponential decay rate of the system's energy by the method of Nakao. For this aim, we need some additional results.

4.4.1 Some results

Lemma 4.4.1. *Let $f \in C^0([a,b],\mathbb{R}_+)$, where $0 < a < b$. Then, there exists a point $x_0 \in [a,b]$ such that*

$$f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \tag{4.19}$$

Proof. 1. By the Weierstrass theorem, f attains its minimum and maximum values on the compact interval [a, b]. Let:

$$f(x_0) = \min_{x\in[a,b]} f(x),$$

2. For all $x \in [a,b]$, it follows that:

$$f(x_0) \le f(x). \tag{4.20}$$

3. Integrating both sides over $[a,b]$:

$$\int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx.$$

4. Since $f(x_0)$ is constant with respect to x, we get:

$$(b-a)f(x_0) \le \int_a^b f(x)\,dx \tag{4.21}$$

5. Dividing through by $(b-a)$

$$f(x_0) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \tag{4.22}$$

Thus, the proof is complete. □

Lemma 4.4.2. *Let* $(\vartheta, \vartheta_t)^\top \in \mathscr{H}$ *be a solution of* (4.1) *such that* $\vartheta_t \neq 0$ *almost everywhere in* $\omega \times (T_0, T_1)$. *We aim to establish the following inequality for some constant* $C_\omega \geq 1$*:*

$$\int_{T_0}^{T_1}\int_\Omega |\vartheta_t(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \leq C_\omega \int_{T_0}^{T_1}\int_\omega |\vartheta_t(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{4.23}$$

Proof. For simplicity, let us define the spacetime domains:

$$\Omega_T = \Omega \times (T_0, T_1), \quad \omega_T = \omega \times (T_0, T_1),$$

and denote the integrals over these domains for a function $f = f(x,t)$ as:

$$\int_{\Omega_T} f\,\mathrm{d}x\,\mathrm{d}s, \quad \int_{\omega_T} f\,\mathrm{d}x\,\mathrm{d}s.$$

Additionally, for any Hilbert space S associated with the norm $\|\cdot\|_S$, $S(\Omega_T)$ denotes the corresponding space over (T_0, T_1) with the norm:

$$\|f\|^2_{S(\Omega_T)} = \int_{T_0}^{T_1} \|f(s)\|_S^2\,\mathrm{d}s.$$

Assume, for the sake of contradiction, that no constant $C > 0$ exists such that inequality (4.23) holds. This implies the existence of a sequence of solutions $\vartheta^{(n)}$ satisfying (4.1) such that:

$$U^{(n)} = (\vartheta^{(n)}, \vartheta_t^{(n)})^\top \in \mathscr{H}, \quad \vartheta_t^{(n)}{/_{\omega_T}} \neq 0,$$

and,

$$\frac{\int_{\Omega_T} |\vartheta_t^{(n)}|^2\,\mathrm{d}x\,\mathrm{d}s}{\int_{\omega_T} |\vartheta_t^{(n)}|^2\,\mathrm{d}x\,\mathrm{d}s} \to +\infty \quad \text{as } n \to +\infty. \tag{4.24}$$

Since $\vartheta^{(n)}$ is bounded in $L^\infty(0,\infty;\mathscr{V}) \cap W^{1,\infty}(0,\infty;L^2(\Omega))$ then there exists a subsequence reindexed in n again by $\vartheta^{(n)}$ such that:

$$\vartheta^{(n)} \to \vartheta \quad \text{weakly in } \mathscr{V}(\Omega_T), \quad \vartheta_t^{(n)} \to \vartheta_t \quad \text{weakly in } L^2(\Omega_T)$$

which implies that

$$\vartheta^{(n)} \to \vartheta \quad \text{strongly in } L^2(\Omega_T).$$

Here, we have two possibilities for ϑ and ϑ_t in Ω_T: ($\vartheta \neq 0$ or $\vartheta = 0$) and ($\vartheta_t \neq 0$ or $\vartheta_t = 0$). Let's analyse each case (ϑ, ϑ_t):

- **For** $(\vartheta, \vartheta_t) \neq 0$ in Ω_T:

  Define the norm of the sequence:

$$\gamma^{(n)} = \left(\int_{\Omega_T} |\vartheta_t^{(n)}|^2 \, \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{2}},$$

and introduce the normalized sequence in $\vartheta_t^{(n)}$:

$$V^{(n)} = (\zeta^{(n)}, \zeta_t^{(n)})^\top, \quad \text{where } \zeta^{(n)} = \frac{\vartheta^{(n)}}{\gamma^{(n)}}.$$

By the linearity of system (4.1), the sequence $\zeta^{(n)}$ also satisfies (4.1). Furthermore,

$$\int_{\Omega_T} |\zeta_t^{(n)}|^2 \, \mathrm{d}x\mathrm{d}s = 1, \quad \forall n \in \mathbb{N}.$$

From (4.24), we also have:

$$\frac{1}{\int_{\omega_T} |\zeta_t^{(n)}|^2 \, \mathrm{d}x \, \mathrm{d}s} \to +\infty \quad \text{when } n \to +\infty,$$

which implies that

$$\int_{\omega_T} |\zeta_t^{(n)}|^2 \, \mathrm{d}x \, \mathrm{d}s \to 0 \quad \text{when } n \to +\infty.$$

As a result, it follows that

$$\zeta_t^{(n)} \to 0 \quad \text{a. e. in } \omega_T. \tag{4.25}$$

Since $\vartheta_t \neq 0$, and $\vartheta^{(n)}$ is bounded in $L^\infty(0,\infty;\mathscr{V}) \cap W^{1,\infty}(0,\infty;L^2(\Omega))$ then the sequence $\zeta^{(n)}$ is bounded in $L^\infty(0,\infty;\mathscr{V}) \cap W^{1,\infty}(0,\infty;L^2(\Omega))$. By the Banach-Alaoglu theorem, there exists a subsequence $V^{(n_k)}$ that converges weakly to $V = (\zeta, \zeta_t)^\top$. Specifically,

$$\zeta^{(n_k)} \to \zeta \quad \text{weakly in } \mathscr{V}(\Omega_T), \quad \zeta_t^{(n_k)} \to \zeta_t \quad \text{weakly in } L^2(\Omega_T)$$

Since $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ then there exists a subsequence indexed in n_k again by $V^{(n_k)} = (\zeta^{(n_k)}, \zeta_t^{(n_k)})^\top$, such that

$$\zeta^{(n_k)} \to \zeta \quad \text{strongly in } L^2(\Omega_T),$$

which means:

$$\zeta^{(n_k)} \to \zeta \quad \text{a. e. in } \Omega_T. \tag{4.26}$$

Now, let's study the system satisfied by $V = (\zeta, \zeta_t)^t$.

Passing to the limit of the system satisfied by $V^{(n_k)}$, and considering (4.25) with the fact that $d(x) = 0$ on $\Omega\backslash\omega$, we obtain:

$$\begin{cases} \zeta_{tt} - \Delta\zeta = 0, & \text{in } \Omega_T, \\ \partial_\nu\zeta = \Delta_T\zeta, & \text{on } \Gamma_{1T}, \\ \zeta = 0, & \text{on } \Gamma_{0T} \\ \zeta_t = 0, & \text{in } \omega_T. \end{cases} \tag{4.27}$$

Taking $z = \zeta_t$ and differentiating with t (4.27) (distributional sense), it follows that:

$$\begin{cases} z_{tt} - \Delta z = 0, & \text{in } \Omega_T \\ z = 0, & \text{in } \omega_T \end{cases}$$

Using standard principle of unique continuation, it follows that $z \equiv 0$ in Ω_T. And as a result, $\zeta_t \equiv 0$ in Ω_T, which contradicts the fact that $\zeta_t \neq 0$. Then the system (4.27) becomes static:

$$\begin{cases} \Delta\zeta = 0, & \text{in } \Omega_T, \\ \partial_\nu\zeta = \Delta_T\zeta, & \text{on } \Gamma_{1T}, \\ \zeta = 0, & \text{on } \Gamma_{0T} \\ \zeta_t = 0, & \text{in } \omega_T. \end{cases}$$

Multiplying the first equation by ζ and integrating over Ω_T, it follows that:

$$\int_{\Omega_T} |\nabla\zeta|^2 \mathrm{d}x\mathrm{d}s + \int_{\Gamma_{1T}} |\nabla_T\zeta|^2 \mathrm{d}x\mathrm{d}s = 0$$

which implies, $\zeta \equiv 0$ a. e. in $\overline{\Omega_T}$ contradicting the fact that $\vartheta \neq 0$.

- **For** $\vartheta \neq 0$ and $\vartheta_t = 0$ in Ω_T:
  As previously, the system satisfied by the weak limit ϑ becomes static. Then it follows that $\vartheta = 0$ a.e. in Ω_T which contradicts the fact that $\vartheta \neq 0$.

- **For** $\vartheta = 0$ and $\vartheta_t \neq 0$ in Ω_T:
  Using the weak continuity property, it follows that the previous normalized sequence $\zeta^{(n)}$ is still bounded in $L^\infty(0, \infty; \mathscr{V}) \cap W^{1,\infty}(0, \infty; L^2(\Omega))$. Then the results obtained remain valid, especially, the weak convergence $\zeta_t^{(n_k)}$ to 0 in $L^2(\Omega_T)$. Which leads to a contradiction with the fact that ϑ_t.

- **For** $(\vartheta, \vartheta_t) = 0$ in Ω_T:

  The verification procedure to determine whether ω satisfies the Geometric Control Condition (GCC in short), the following inequality (uniform in n) is obtained as in [17] for $T = T_1 - T_0$ large enough:

$$\xi_u(t) \leq C\left(\int_{\omega_T} |u_t|^2 \mathrm{d}\omega_T + \int_{\Omega_T} |u|^2 \mathrm{d}\Omega_T\right), \quad \forall t \in (T_1, T_0),$$

  with $C > 0$, u satisfies (4.1) and ξ_u its energy.

  This immediately implies that the energy $\xi_\eta^{(n)}$ released by the normalized sequence $\eta^{(n)} = \frac{\vartheta^{(n)}}{\|\vartheta^{(n)}\|}$ in $L^2(\Omega_T)$ satisfies (4.24) and:

$$\xi_\eta^{(n)}(t) \leq C\left(\int_{\omega_T} |\eta_t^{(n)}|^2 \mathrm{d}\omega_T + 1\right), \quad \forall t \in (T_0, T_1).$$

  Using (4.24), it follows that $\eta^{(n)}$ is still bounded in $L^\infty((T_0, T_1); \mathscr{V}) \cap W^{1,\infty}((T_0, T_1); L^2(\Omega))$. Using the same procedure as previously, one can find a subsequence $\eta^{(n_k)}$ which finally converge strongly to 0 in $L^2(\Omega_T)$. Thus contradicting the normalized aspect of $\eta^{(n)}$ in $L^2(\Omega_T)$.

Hence, the inequality (4.23) must hold for some $C_\omega > 0$.
The proof is complete. □

4.4.2 Exponential Decay result

First, let's write the energy ξ with all its terms. From (4.28), ξ is defined as:

$$\xi(t) = \frac{1}{2}\left(\int_\Omega |\nabla\vartheta|^2 \, \mathrm{d}x + \int_{\Gamma_1} |\nabla_T\vartheta|^2 \, \mathrm{d}\Gamma_1 + \int_\Omega |\vartheta_t|^2 \, \mathrm{d}x\right), \quad \forall t \geq 0. \tag{4.28}$$

Lemma 4.4.3. *Let $\varphi(t)$ be a bounded and positive function defined on* $[0, +\infty[$. *Suppose $\varphi(t)$ satisfies the inequality:*

$$\sup_{s\in[t,t+1]} \varphi(s) \leq C_0(\varphi(t) - \varphi(t+1)), \quad \forall t \geq 0. \tag{4.29}$$

for some positive constant C_0.
Under these conditions, there exist positive constants M and α such that

$$\varphi(t) \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

Proof. The proof follows directly from the methodology outlined in reference [75, lemma 3]. This result establishes the exponential decay of $\varphi(t)$ under the given assumptions. □

Theorem 4.4.4. *Let* $(\vartheta(x,t),\ \vartheta_t(x,t))^\top \in \mathscr{H}$ *be a solution of the system* (4.1)*. Then, the solution satisfies the following energy inequality:*

$$\xi(t) \leq Me^{-\alpha t}, \quad \forall t \geq 0, \tag{4.30}$$

where M *and* α *are positive constants, and* $\xi(t)$ *represents the total energy defined in* (4.28).

Proof. Integrating (4.15) member by member over $[t, t+1]$, we get

$$\xi(t) - \xi(t+1) = \int_t^{t+1} \int_\Omega d(x) |\vartheta_t(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \ := K^2(t). \tag{4.31}$$

The approach adopted aligns with the methods described in [12, 75, 78]. We define the function F as follows:

$$F(t) = d_0 \|\vartheta_t(t)\|^2_{L^2(\omega)}. \tag{4.32}$$

It is clear that F is a non-negative function and continuous over $[t, t+1]$ for $t \geq 0$. By applying Lemma **4.4.1** to the function F over $[t, t+\frac{1}{4}]$ and $[t+\frac{3}{4}, t+1]$, respectively, we obtain the following inequalities for some points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$:

$$F(t_1) \leq 4 \int_t^{t+\frac{1}{4}} F(s)\,ds \quad \text{and} \quad F(t_2) \leq 4 \int_{t+\frac{3}{4}}^{t+1} F(s)\,ds.$$

From these, we bound $F(t_1)$ and $F(t_2)$ by integrals over $[t, t+1]$:

$$F(t_1) \leq 4 \int_t^{t+1} F(s)\,ds \quad \text{and} \quad F(t_2) \leq 4 \int_t^{t+1} F(s)\,ds.$$

Substituting F from equation (4.32) and simplifying, we find that:

$$\|\vartheta_t(x,t_i)\|^2_{L^2(\omega)} = \int_\omega |\vartheta_t(x,t_i)|^2 \,\mathrm{d}x \leq \frac{4}{d_0} K^2(t), \quad \forall i \in \{1,2\}. \tag{4.33}$$

Multiplying $(4.1)_1$ by ϑ and integrating by parts over Ω, we obtain

$$\int_\Omega \vartheta_{tt}\vartheta \,\mathrm{d}x - \int_\Omega \vartheta\Delta\vartheta \,\mathrm{d}x + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0. \tag{4.34}$$

Using Green's formula and the fact that $\vartheta_{|\Gamma_0} = 0$ and $\partial_\nu\vartheta_{|\Gamma_1} = \Delta_T\vartheta_{|\Gamma_1}$, we get

$$\begin{aligned}\int_\Omega \vartheta\Delta\vartheta \,\mathrm{d}x &= -\int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x + \int_{\Gamma_1} \vartheta\partial_\nu\vartheta \,\mathrm{d}\Gamma_1 \\ &= -\int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x + \int_{\Gamma_1} \vartheta\Delta_T\vartheta \,\mathrm{d}\Gamma_1 \\ &= -\int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x - \int_{\Gamma_1} |\nabla_T\vartheta|^2 \,\mathrm{d}\Gamma_1.\end{aligned} \tag{4.35}$$

Using the result (4.35) and the fact that $\vartheta_{tt}\vartheta = \frac{d}{dt}(\vartheta_t\vartheta) - |\vartheta_t|^2$ in (4.34), we get

$$\frac{d}{dt}\left(\int_\Omega \vartheta_t\vartheta \,\mathrm{d}x\right) - \int_\Omega |\vartheta_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x + \int_{\Gamma_1} |\nabla_T\vartheta|^2 \,\mathrm{d}\Gamma_1 + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0. \tag{4.36}$$

Integrating (4.36) over $[t_1, t_2]$, we get

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)}\right) \mathrm{d}s = I_1 - I_2 + \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\Omega)} \,\mathrm{d}s - J. \tag{4.37}$$

where

$$\begin{aligned} I_i &= \int_\Omega \vartheta_t(x,t_i)\vartheta(x,t_i)\,\mathrm{d}x, \quad \forall i \in \{1,2\}, \\ J &= \int_{t_1}^{t_2}\int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.38}$$

Using the Cauchy-Schwarz inequality and (4.23), we get

$$|I_i| \le \|\vartheta_t(t_i)\|_{L^2(\Omega)}\|\vartheta(t_i)\|_{L^2(\Omega)} \le C_\omega\|\vartheta_t(t_i)\|_{L^2(\omega)}\|\vartheta(t_i)\|_{L^2(\Omega)}, \quad \forall i \in \{1,2\}. \tag{4.39}$$

Using the Cauchy-Schwarz inequality, we find:

$$|J| \le d_0 \int_{t_1}^{t_2} \|\vartheta_t\|_{L^2(\omega)}\|\vartheta\|_{L^2(\omega)} \,\mathrm{d}s.$$

Since $ab \le \frac{a^2+b^2}{2}$ for $a, b \in \mathbb{R}$,, we get

$$\begin{aligned} |J| &\le \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s + \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta\|^2_{L^2(\omega)} \,\mathrm{d}s \\ &\le \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s + \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta\|^2_{L^2(\Omega)} \,\mathrm{d}s. \end{aligned}$$

Using the Poincaré inequality on the second integral, we get

$$|J| \le \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s + \frac{d_0 c_p}{2}\int_{t_1}^{t_2} \|\nabla\vartheta\|^2_{L^2(\Omega)} \,\mathrm{d}s, \tag{4.40}$$

where $c_p > 0$ is Poincaré's constant.
Combining (4.40) and (4.39) in (4.37), we obtain

$$\begin{aligned} \left(1 - \frac{c_p d_0}{2}\right)\int_{t_1}^{t_2} \|\nabla\vartheta\|^2_{L^2(\Omega)} \,\mathrm{d}s + \int_{t_1}^{t_2} \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} \,\mathrm{d}s \le \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\Omega)} \,\mathrm{d}s + C_\omega S \\ + \frac{d_0}{2}\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s \end{aligned} \tag{4.41}$$

where

$$S = \sum_{i=1}^{2} \|\vartheta_t(t_i)\|_{L^2(\omega)} \|\vartheta(t_i)\|_{L^2(\Omega)}. \tag{4.42}$$

Using (4.23) on the right-hand side of (4.41) and choosing $d_0 < \frac{2}{c_p}$, we get

$$\left(1 - \frac{c_p d_0}{2}\right) \int_{t_1}^{t_2} \|\nabla\vartheta\|^2_{L^2(\Omega)} \,\mathrm{d}s + \int_{t_1}^{t_2} \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} \,\mathrm{d}s \le C_\omega \left(\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s + S \right) + \frac{d_0}{2} \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s.$$

Then

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} \right) \mathrm{d}s \le l \left(S + d_0 \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s \right), \tag{4.43}$$

where $l = \frac{2}{2 - c_p d_0} \max\left(C_\omega, \frac{1}{2}\right) = \frac{2C_\omega}{2 - c_p d_0}$.
Using the definition of K in (4.31), it follows that

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} \right) \mathrm{d}s \le l \left(S + K^2(t) \right). \tag{4.44}$$

Using (4.33) and the Poincaré inequality, we have

$$\begin{aligned} |S| &\le \frac{2c_p}{\sqrt{d_0}} K(t) \left(\|\nabla\vartheta(t_1)\|_{L^2(\Omega)} + \|\nabla\vartheta(t_2)\|_{L^2(\Omega)} \right) \\ &\le \frac{4c_p}{\sqrt{d_0}} K(t) \sup_{s\in[t,t+1]} \|\nabla\vartheta(s)\|_{L^2(\Omega)}, \end{aligned}$$

We set

$$\lambda = \frac{4c_p}{\sqrt{d_0}} l \quad \text{and} \quad \tilde{K}^2(t) = \lambda \left[K(t) \sup_{s\in[t,t+1]} \|\nabla\vartheta(s)\|_{L^2(\Omega)} + K^2(t) \right].$$

Then

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} \right) \mathrm{d}s \le \tilde{K}^2(t). \tag{4.45}$$

Adding $\int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\Omega)} \,\mathrm{d}s$ to both sides of (4.45), we get

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} + \|\vartheta_t\|^2_{L^2(\Omega)} \right) \mathrm{d}s \le \tilde{K}^2(t) + \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\Omega)} \,\mathrm{d}s. \tag{4.46}$$

Using (4.23) on the right-hand side of (4.46), it follows that

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)} + \|\vartheta_t\|^2_{L^2(\Omega)} \right) \mathrm{d}s \le \tilde{K}^2(t) + C_\omega \int_{t_1}^{t_2} \|\vartheta_t\|^2_{L^2(\omega)} \,\mathrm{d}s. \tag{4.47}$$

Using (4.31) on the right-hand side of (4.47), we get

$$\int_{t_1}^{t_2}\left(\|\nabla\vartheta\|^2_{L^2(\Omega)}+\|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)}+\|\vartheta_t\|^2_{L^2(\Omega)}\right)\mathrm{d}s\le\rho(\tilde{K}^2(t)+K^2(t)),\tag{4.48}$$

where $\rho=\max\left(1,\frac{4C_\omega}{d_0}\right)$.
Using Lemma **4.4.1**, it follows that there exists $t_0\in[t_1,t_2]$ such that

$$\|\nabla\vartheta(t_0)\|^2_{L^2(\Omega)}+\|\nabla_T\vartheta(t_0)\|^2_{L^2(\Gamma_1)}+\|\vartheta_t(t_0)\|^2_{L^2(\Omega)}\le\frac{1}{t_2-t_1}\rho(\tilde{K}^2(t)+K^2(t)).$$

Since $t_1\in[t,t+\frac{1}{4}]$, and $t_2\in[t+\frac{3}{4},t+1]$, then

$$\frac{1}{t_2-t_1}\in[1,2].$$

This implies that

$$\xi(t_0)\le2\rho(\tilde{K}^2(t)+K^2(t)).\tag{4.49}$$

Next, integrating by parts over $[t,t_0]$ in equation (4.15), we obtain

$$\xi(t)=\xi(t_0)+\int_t^{t_0}\int_\Omega d(x)|\vartheta_t|^2\,\mathrm{d}x\,\mathrm{d}s.\tag{4.50}$$

Using (4.49) and (4.31), we get

$$\begin{aligned}\sup_{s\in[t,t+1]}\xi(s)&\le(1+2\rho)K^2(t)+2\rho\lambda\left[K(t)\sup_{s\in[t,t+1]}\left(\|\nabla\vartheta(s)\|_{L^2(\Omega)}\right)+K^2(t)\right]\\&\le[1+2\rho(1+\lambda)]K^2(t)+2\rho\lambda K(t)\sup_{s\in[t,t+1]}\left(\sqrt{2\xi(s)}\right)\\&\le[1+2\rho(1+\lambda)]K^2(t)+4\lambda^2\rho^2K^2(t)+\frac{1}{2}\sup_{s\in[t,t+1]}\xi(s).\end{aligned}\tag{4.51}$$

Thus,

$$\frac{1}{2}\sup_{s\in[t,t+1]}\xi(s)\le[1+2\rho(1+\lambda)+4\rho^2\lambda^2]K^2(t),$$

which immediately yields

$$\sup_{s\in[t,t+1]}\xi(s)\le C(\xi(t)-\xi(t+1)),\quad\text{where }C=2[1+2\rho(1+\lambda)+4\rho^2\lambda^2].\tag{4.52}$$

Finally, the result stated in Lemma 4.4.3, as referenced in (4.29), is obtained. Consequently, the problem (4.1) under study, satisfies the exponential stabilization estimate given in (4.30). □

Now, let's give the influence of d_0, c_p, and C_ω on the exponential decay rate.

4.4.3 Impact of parameters

First all, we need to express the result (4.52) as the condition of Proposition 2.6.9 in previous **Chapter** 2.

From (4.52), we have:

$$\sup_{s\in[t,t+1]} \xi(s) \leq C(\xi(t) - \xi(t+1)), \quad C > 1.$$

Since ξ is continuously dissipative then

$$\sup_{s\in[t,t+1]} \xi(s) = \xi(t), \forall t \geq 0.$$

Consequently (4.52) becomes,

$$\xi(t) \leq C(\xi(t) - \xi(t+1)),$$

which implies that

$$\xi(t+1) \leq (1-k)\xi(t).$$

where $k = \frac{1}{C} \in]0;1[$.

Using Proposition 2.6.9, it follows that:

$$\xi(t) \leq \frac{C}{C-1} e^{log\left(1-\frac{1}{C}\right)t}\xi(0), \quad \forall t > 1. \tag{4.53}$$

Now, let's define $\rho = \rho(s,h)$ and $\lambda = \lambda(s,h)$ such that for all $0 < h < \frac{2}{c_p}$ and $s \geq 1$:

$$\rho(s,h) = max\left(1, \frac{4s}{h}\right), \quad \lambda(s,h) = \frac{8c_p s}{\sqrt{h}(2 - c_p h)}. \tag{4.54}$$

Proposition 4.4.5. *The functions ρ and λ satisfy:*

i. If $0 < h < \frac{2}{c_p}$ and $s \geq 1$, then ρ and λ are nonzero and bounded.

ii. If $h \underset{h<\frac{2}{c_p}}{\longrightarrow} \frac{2}{c_p}$ then $\lambda(s,h) \to +\infty$, forall $s \geq 1$. Moreover, if $c_p \to 0^+$ then h may tend to ∞.

iii. If $h \to 0^+$ then $\rho(s,h) \to +\infty$, for a given $s \geq 1$. Moreover, if $c_p \to +\infty$ then $h \to 0^+$.

iv. If $s \to +\infty$ then $\rho(s,h) \to +\infty$ for a given $h < \frac{2}{c_p}$.

Proof. All the assertions (i-iv) are obvious. □

Corollary 4.4.6. *Let* $(\vartheta, \vartheta_t)^\top \in \mathscr{H}$ *be a solution of system* (4.1)*. Then the exponential decay rate* (4.53) *gets lost when:*

a. $d_0 \underset{d_0<\frac{2}{c_p}}{\longrightarrow} \frac{2}{c_p}$ *or* $d_0 \to 0^+$ *or* $c_p \to +\infty$

b. $C_\omega \to +\infty$.

Proof. Define $\rho = \rho(d_0, C_\omega)$ and $\lambda = \lambda(d_0, C_\omega)$ given in (4.54).

It is obvious that:

$$\frac{C}{C-1} e^{log\left(1-\frac{1}{C}\right)t} \longrightarrow 1, \quad \text{when } C \to +\infty.$$

Since $C = C(\rho, \lambda) = 2[1 + 2\rho(1+\lambda) + 4\rho^2\lambda^2]$, then

$$C \to +\infty, \quad \text{when } \rho \to +\infty \text{ or } \lambda \to +\infty.$$

Consequently, by the use of Proposition 4.4.5 combined with (4.53), it follows that:

$$\xi(t) \le \frac{C}{C-1} e^{log\left(1-\frac{1}{C}\right)t} \xi(0) \to \xi(0)$$

when the conditions (a) and (b) hold.
The proof is complete. □

Remark 4.4.7. *Due to the lost of the exponential decay rate* (4.53) *when* d_0 *is closed to* 0 *and* $\frac{2}{c_p}$*, there might be a threshold for which the rate* (4.53) *decays more rapidly than the other choices of* d_0.

During the research time, this chapter was conducted as a test for Nakao's technique. Then, this chapter do not illustrate this fact of threshold for $d_0 < \frac{2}{c_p}$ *(**see next chapter**).*

Remark 4.4.8. *From physical perspective, the corollary* 4.4.6 *indicates the lost of the system's exponential stability when :*

a. the dissipation coefficient d_0 *of the damper exceeds/or it is too weak/or if the medium* Ω *is very small.*

b. the damped region ω *is very small in* Ω *or the energy released by the wave's speed in* Ω *is infinitely larger than that in* ω.

4.5 Numerical simulation

To perform a numerical analysis of the exponential stability discussed in the preceding part in one dimension, we will examine the two-dimensional problem presented below, which is connected with equation (4.1). By applying Fourier analysis, we may express the problem in one dimension.

$$\begin{cases} \vartheta_{tt} - \vartheta_{xx} - \vartheta_{yy} + d(x,y)\vartheta_t = 0, & \text{in } \Omega \times \mathbb{R}_+ \\ \partial_\nu \vartheta = \Delta_T \vartheta, & \text{on } \Gamma_1 \times \mathbb{R}_+ \\ \vartheta = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+ \\ \vartheta(.,0) = \vartheta_0, \vartheta_t(.,0) = \vartheta_1, & \text{in } \Omega \end{cases} \tag{4.55}$$

where $\vartheta = \vartheta((x,y),t)$, $\Omega = \{u = (x,y) \in \mathbb{R}^2 \ : \ 0 < r_0 < |u| = x^2 + y^2 < 1\} \subset \mathbb{R}^2$ is an open domain with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$

$$\Gamma_0 = \{u \in \mathbb{R}^2 | \, |u| = r_0\}, \quad \Gamma_1 = \{u \in \mathbb{R}^2 | \, |u| = 1\}$$

We will use the Fourier series expansion to transform (4.55) into a one-dimensional system. This is possible because the boundary Γ_1 has both normal and tangential derivatives. Assume that ϑ is written as follows:

$$\vartheta((x,y),t) = \sum_{k\geq 0} \vartheta^{(k)}(x,t)\cos(Ly), \quad \forall ((x,y),t) \in \Omega \times \mathbb{R}_+. \tag{4.56}$$

where $\vartheta^{(k)}$ are the coefficients of the Fourier series and $L \in \mathbb{R}_+^*$ is a constant.
Let's inject (4.56) in (4.55), it is clear that, on Ω we obtain

$$\begin{aligned} &\sum_{k\geq 0} \vartheta_{tt}^{(k)}(x,t)cos(Ly) - \sum_{k\geq 0} \vartheta_{xx}^{(k)}(x,t)cos(Ly) \\ &+ L^2 \sum_{k\geq 0} \vartheta^{(k)}(x,t)cos(Ly) + d(x,y) \sum_{k\geq 0} \vartheta_t^{(k)}(x,t)cos(Ly) = 0 \end{aligned} \tag{4.57}$$

which implies

$$\vartheta_{tt}^{(k)}(x,t) - \vartheta_{xx}^{(k)}(x,t) + L^2\vartheta^{(k)}(x,t) + d(x,y)\vartheta_t^{(k)}(x,t) = 0. \tag{4.58}$$

Since $\partial_\nu f = \nabla f.\nu$, $f \in C^1$ and (.) is the standard inner product of $\mathbb{R}^2$, then

$$\partial_\nu \vartheta((x,y),t) = x\vartheta_x((x,y),t) + y\vartheta_y((x,y),t), \quad \forall (x,y) \in \Gamma_1, \ t \geq 0 \tag{4.59}$$

Then using (4.56), we get

$$\partial_\nu \vartheta((x,y),t) = \sum_{k\geq 0} \left(x\vartheta_x^{(k)}(x,t)cos(Ly) - Ly\vartheta^{(k)}(x,t)sin(Ly) \right). \tag{4.60}$$

We also have

$$\begin{aligned}\Delta_T\vartheta((x,y),t) &= \frac{\partial^2}{\partial y^2}\left(\sum_{k\geq 0}\vartheta^{(k)}(x,t)cos(Ly)\right)\\ &= -L^2\sum_{k\geq 0}\vartheta^{(k)}(x,t)cos(Ly)\end{aligned} \tag{4.61}$$

Thus, the boundary condition $\partial_\nu\vartheta = \Delta_T\vartheta$ over Γ_1, is

$$\sum_{k\geq 0}\left[\left(x\vartheta_x^{(k)}(x,t) + L^2\vartheta^{(k)}(x,t)\right)cos(Ly) - Ly\vartheta^{(k)}(x,t)sin(Ly)\right] = 0 \tag{4.62}$$

We therefore chose $y = 0$ to have a one dimensional case , and the only possible values for x are 1 and -1 on Γ_1 since $x^2 + y^2 = 1$. To simplify the numerical study, we limit ourselves to $x = 1$ only. Then (4.62) becomes

$$\sum_{k\geq 0}\left[\left(\vartheta_x^{(k)}(1,t) + L^2\vartheta^{(k)}(1,t)\right)\right] = 0, \tag{4.63}$$

Which implies

$$\vartheta_x^{(k)}(1,t) = -L^2\vartheta^{(k)}(1,t) \tag{4.64}$$

Moreover, the Dirichlet condition imposes that $\vartheta^{(k)}(r_0,t) = 0$.

This last condition combined with (4.58) and (4.64) allows us to deduce the following one-dimensional problem

$$\begin{cases}\vartheta_{tt}^{(k)}(x,t) - \vartheta_{xx}^{(k)}(x,t) + L^2\vartheta^{(k)}(x,t) + d(x)\vartheta_t^{(k)}(x,t) = 0, & (x,t)\in\Omega_x\times\mathbb{R}_+\\ \vartheta_x^{(k)}(1,t) = -L^2\vartheta^{(k)}(1,t), & t\in\mathbb{R}_+\\ \vartheta^{(k)}(r_0,t) = 0, & t\in\mathbb{R}_+\\ \vartheta^{(k)}(x,0) = \vartheta_0^{(k)}(x), \vartheta_t^{(k)}(x,0) = \vartheta_1^{(k)}(x), & x\in\Omega_x\end{cases} \tag{4.65}$$

where $\Omega_x = (r_0, 1)$. For simplicity, we will drop the index k moving forward.

The numerical scheme associated with (4.65) in finite difference is:

$$\begin{cases}\dfrac{\vartheta_\imath^{\jmath+1} - 2\vartheta_\imath^\jmath + \vartheta_\imath^{\jmath-1}}{(\Delta t)^2} - \dfrac{\vartheta_{\imath+1}^\jmath - 2\vartheta_\imath^\jmath + \vartheta_{\imath-1}^\jmath}{(\Delta x)^2} + L^2\vartheta_\imath^\jmath + d_\imath\dfrac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1}}{2\Delta t} = 0\\ \vartheta_{N_x+1}^\jmath = -\dfrac{1}{L^2\Delta x}\left(\vartheta_{N_x+1}^\jmath - \vartheta_{N_x}^\jmath\right)\\ \vartheta_{0,j} = 0\\ \vartheta_{\imath,0} = \vartheta_0(x_\imath), \quad \vartheta_{\imath,1} = \vartheta_{\imath,0} + \Delta t\vartheta_1(x_\imath)\end{cases} \tag{4.66}$$

for all $1 \le \imath \le N_x$ and $1 \le \jmath \le N_t$ such that $N_x + 2 \in \mathbb{N}$ (resp. $N_t + 2 \in \mathbb{N}$) equidistant points are chosen in $\overline{\Omega}_x$ (resp. the time interval $[0;T]$, $T > 0$) as follows:

$$\begin{cases} x_0 = r_0 \\ \Delta x = \dfrac{1 - r_0}{N_x + 1} \\ x_\imath = x_0 + \imath \Delta x, \quad \forall \imath \in \{0, 1, \dots, N_x\} \\ x_{N_x+1} = 1 \end{cases}, \quad \text{and} \quad \begin{cases} T_0 = 0 \\ \Delta t = \dfrac{T}{N_t + 1} \\ T_\jmath = T_0 + \jmath \Delta t, \quad \forall \jmath \in \{0, 1, \dots, N_t\} \\ T_{N_t+1} = T \end{cases} \tag{4.67}$$

and $\vartheta(x_\imath, t_\jmath) = \vartheta_\imath^\jmath$, $d(x_\imath) = d_\imath$.

The full energy expression for the system (4.66) is given by:

$$\begin{aligned} \xi_\jmath = {} & \frac{1}{2L^2} \frac{\vartheta_{N_x+1}^{\jmath+1} - \vartheta_{N_x}^{\jmath+1}}{\Delta x} \frac{\vartheta_{N_x+1}^{\jmath} - \vartheta_{N_x}^{\jmath}}{\Delta x} \\ & + \frac{L^2}{2} \Delta x \sum_{\imath=0}^{N_x} \vartheta_\imath^{\jmath+1} \vartheta_\imath^\jmath \\ & + \frac{\Delta x}{2} \sum_{\imath=0}^{N_x} \left(\frac{\vartheta_{\imath+1}^{\jmath+1} - \vartheta_\imath^{\jmath+1}}{\Delta x} \frac{\vartheta_{\imath+1}^{\jmath} - \vartheta_\imath^{\jmath}}{\Delta x} + \left| \frac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^\jmath}{\Delta t} \right|^2 \right) \end{aligned} \tag{4.68}$$

Proposition 4.5.1. *If $\vartheta_\imath^\jmath$ is a solution of the numerical scheme* (4.66)*, then its energy $\xi_\jmath$ is dissipative for all $\Delta x, \Delta t \in (0,1)$ at time T_j, and satisfies*

$$\frac{\xi_\jmath - \xi_{\jmath-1}}{\Delta t} = -\Delta x \sum_{\imath=0}^{N_x} d_\imath \left| \frac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1}}{2\Delta t} \right|^2. \tag{4.69}$$

Proof. Multiplying the first equation of the system (4.66) by $\frac{\Delta x}{2\Delta t} \left(\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1} \right)$ and summing from $i = 1$ to $i = N_x$, we obtain the following expression:

$$\begin{aligned} & \frac{\Delta x}{2(\Delta t)^3} \sum_{\imath=1}^{N_x} \left(\vartheta_\imath^{\jmath+1} - 2\vartheta_\imath^\jmath + \vartheta_\imath^{\jmath-1} \right) \left(\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1} \right) \\ & - \frac{\Delta x}{2\Delta t (\Delta x)^2} \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^\jmath - 2\vartheta_\imath^\jmath + \vartheta_{\imath-1}^\jmath \right) \left(\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1} \right) \\ & + L^2 \frac{\Delta x}{2\Delta t} \sum_{\imath=1}^{N_x} \vartheta_\imath^\jmath \left(\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1} \right) \\ & + \Delta x \sum_{\imath=1}^{N_x} d_\imath \left| \frac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^\jmath}{2\Delta t} \right|^2 = 0. \end{aligned} \tag{4.70}$$

- As $\vartheta_{0,J} = 0$ and using the identity $(a+c-2b)(a-c) = (a-b)^2 - (b-c)^2$ for all $a, b, c \in \mathbb{R}$, the first sum can be written as:

$$\sum_{\imath=1}^{N_x} \left(\vartheta_\imath^{J+1} - 2\vartheta_\imath^J + \vartheta_\imath^{J-1}\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right) = \sum_{\imath=1}^{N_x} \left|\vartheta_\imath^{J+1} - \vartheta_\imath^J\right|^2 - \sum_{\imath=1}^{N_x} \left|\vartheta_\imath^J - \vartheta_\imath^{J-1}\right|^2. \tag{4.71}$$

- For the second sum, using the fact that $\vartheta_{0,J} = 0$, we get:

$$\begin{aligned}\sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - 2\vartheta_\imath^J + \vartheta_{\imath-1}^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right) &= \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right)\\ &+ \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath-1}^J - \vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right)\\ &= \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right)\\ &- \sum_{\imath=0}^{N_x-1} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_{\imath+1}^{J+1} - \vartheta_{\imath+1}^{J-1}\right).\end{aligned}$$

This simplifies to:

$$\begin{aligned}\sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - 2\vartheta_\imath^J + \vartheta_{\imath-1}^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right) &= \left(\vartheta_{N_x+1}^J - \vartheta_{N_x}^J\right)\left(\vartheta_{N_x+1}^{J+1} - \vartheta_{N_x+1}^{J-1}\right)\\ &+ \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right)\\ &- \sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_{\imath+1}^{J+1} - \vartheta_{\imath+1}^{J-1}\right).\end{aligned}$$

Using the boundary condition, we get:

$$\vartheta_{N_x+1}^{J+1} - \vartheta_{N_x+1}^{J-1} = -\frac{1}{L^2\Delta x}\left[\left(\vartheta_{N_x+1}^{J+1} - \vartheta_{N_x}^{J+1}\right) - \left(\vartheta_{N_x+1}^{J-1} - \vartheta_{N_x}^{J-1}\right)\right].$$

Therefore, we have:

$$\begin{aligned}\sum_{\imath=1}^{N_x} \left(\vartheta_{\imath+1}^J - 2\vartheta_\imath^J + \vartheta_{\imath-1}^J\right)\left(\vartheta_\imath^{J+1} - \vartheta_\imath^{J-1}\right) &= -\frac{1}{L^2\Delta x}\left(\vartheta_{N_x+1}^{J+1} - \vartheta_{N_x}^{J+1}\right)\left(\vartheta_{N_x+1}^J - \vartheta_{N_x}^J\right)\\ &- \sum_{\imath=0}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_{\imath+1}^{J+1} - \vartheta_\imath^{J+1}\right)\\ &+ \frac{1}{L^2\Delta x}\left(\vartheta_{N_x+1}^J - \vartheta_{N_x}^J\right)\left(\vartheta_{N_x+1}^{J-1} - \vartheta_{N_x}^{J-1}\right)\\ &+ \sum_{\imath=0}^{N_x} \left(\vartheta_{\imath+1}^J - \vartheta_\imath^J\right)\left(\vartheta_{\imath+1}^{J-1} - \vartheta_\imath^{J-1}\right).\end{aligned}$$

Replacing the first and second sums by their new expressions in (4.70), we obtain

$$\begin{aligned}
&\frac{1}{2L^2\Delta t}\frac{\vartheta^{J+1}_{N_x+1}-\vartheta^{J+1}_{N_x}}{\Delta x}\frac{\vartheta^{J}_{N_x+1}-\vartheta^{J}_{N_x}}{\Delta x}+L^2\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{N_x}\vartheta_\imath^{J+1}\vartheta_\imath^{J}\\
&+\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{N_x}\left(\frac{\vartheta^{J+1}_{\imath+1}-\vartheta^{J+1}_{\imath}}{\Delta x}\frac{\vartheta^{J}_{\imath+1}-\vartheta^{J}_{\imath}}{\Delta x}+\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J}_{\imath}}{\Delta t}\right|^2\right)\\
&-\frac{1}{2L^2\Delta t}\frac{\vartheta^{J}_{N_x+1}-\vartheta^{J}_{N_x}}{\Delta x}\frac{\vartheta^{J-1}_{N_x+1}-\vartheta^{J-1}_{N_x}}{\Delta x}-L^2\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{N_x}\vartheta_\imath^{J}\vartheta_\imath^{J-1}\\
&-\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{N_x}\left(\frac{\vartheta^{J}_{\imath+1}-\vartheta^{J}_{\imath}}{\Delta x}\frac{\vartheta^{J-1}_{\imath+1}-\vartheta^{J-1}_{\imath}}{\Delta x}+\left|\frac{\vartheta^{J}_{\imath}-\vartheta^{J-1}_{\imath}}{\Delta t}\right|^2\right)\\
&=-\Delta x\sum_{\imath=1}^{N_x}d_\imath\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J-1}_{\imath}}{2\Delta t}\right|^2
\end{aligned}\tag{4.72}$$

Which is equivalent to

$$\frac{\xi_J-\xi_{J-1}}{\Delta t}=-\Delta x\sum_{\imath=1}^{N_x}d_\imath\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J-1}_{\imath}}{2\Delta t}\right|^2$$

The proof is complete. □

4.5.1 Test Problems

In this section, we present a series of one-dimensional tests designed to deepen and extend the analysis previously discussed. These examples aim to demonstrate practical applications and validate the theoretical concepts addressed.

We consider the one-dimensional problem (5.48) with various initial conditions over $\Omega_x=\left(\frac{1}{2},1\right)$, $\omega_x=(\frac{3}{4},1)$, $L=0.5$, and $d(x)\geq 0.2$ in ω_x:

1.

$$\vartheta_0(x)=\sin\left(\frac{2\pi x}{L}\right),\quad \vartheta_1(x)=0\quad\text{in }\Omega_x\tag{4.73}$$

2.

$$\vartheta_0(x)=\begin{cases}(x-1)e^{Lx}, & \text{if } x\in]\frac{1}{2},1]\\ 0, & \text{if } x=\frac{1}{2}\end{cases},\quad \vartheta_1(x)=\begin{cases}\frac{(x-1)^2}{L}e^{Lx} & \text{if } x\in]\frac{1}{2},1]\\ 0, & \text{if } x=\frac{1}{2}.\end{cases}\tag{4.74}$$

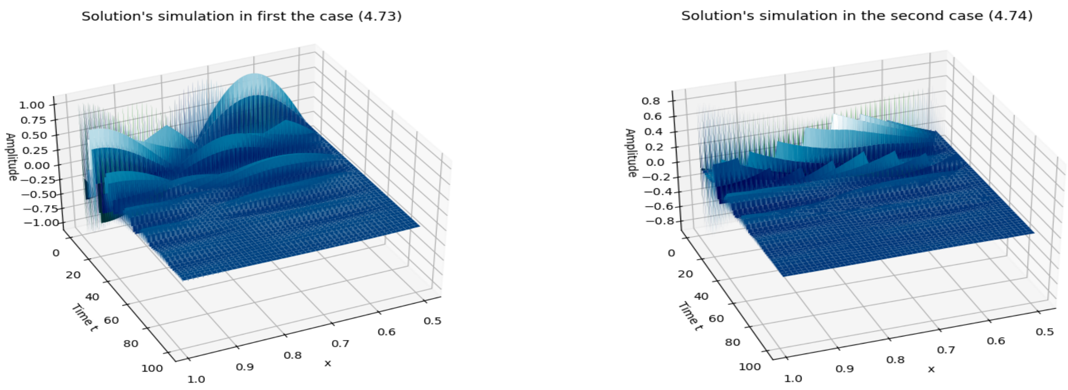

Figure 4.1: 1st and 2nd solutions for $d_0 = 0.2$ (Global views)

As can see here, **Fig. 4.1** shows a more flexibility over the line $x = 1$ than that of $x = 0.5$. This illustrate in general, how the phenomenon happens over the boundary Γ_1 and Γ_0.

Now, let's illustrate the impact of d_0.

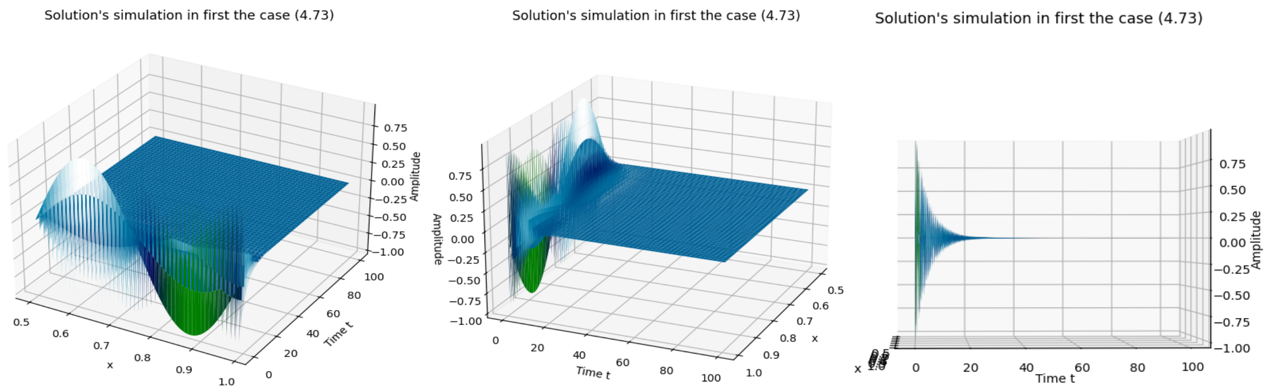

Figure 4.2: 1st solution for $d_0 = 1$ (Global and Time views)

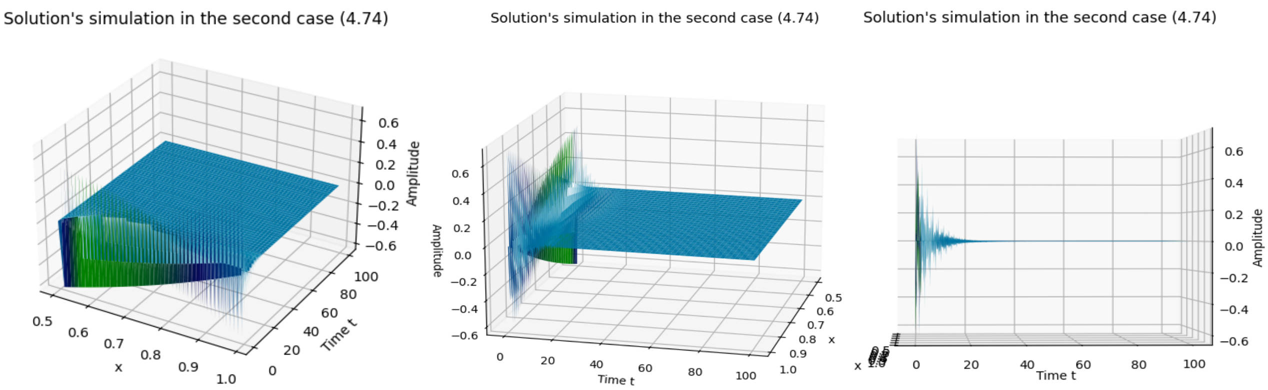

Figure 4.3: 2nd solution for $d_0 = 1$ (Global and Time views)

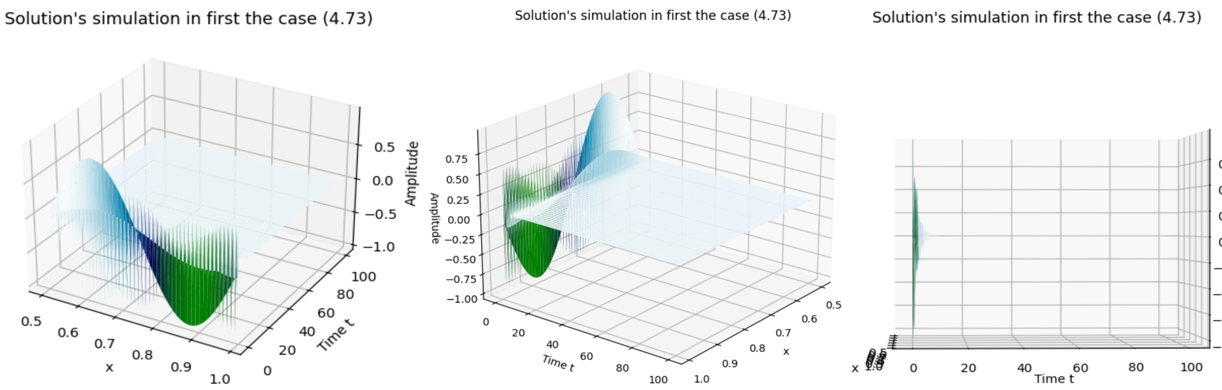

Figure 4.4: 1^{st} solution for $d_0 = 3$ (Global and Time views)

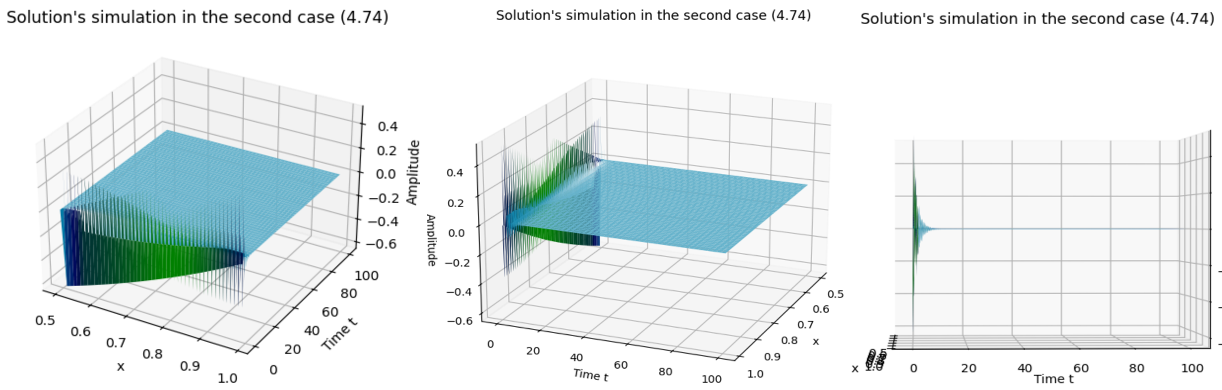

Figure 4.5: 2^{nd} solution for $d_0 = 3$ (Global and Time views)

4.5.2 Global and Time views analysis

In this study on system dynamics in response to different levels of damping, characterized by $d_0 = 1$ and $d_0 = 3$, **Fig.** $(4.2 - 4.5)$ demonstrate the impact of damping parameters on energy dissipation and oscillatory behavior. With moderate damping ($d_0 = 1$), we observe a gradual decline in oscillations, where the amplitude decreases exponentially but remains evident over an extended period (see **Fig.** 4.2 and **Fig.** 4.3). In contrast, more intense damping ($d_0 = 3$) leads to a rapid and significant reduction in oscillations (see **Fig.** 4.4 and **Fig.** 4.5). This regime quickly diminishes oscillations, accelerating the convergence towards a stable state and reducing periods of unwanted oscillations.

Moreover, **Figures** 4.4 and **Fig.** 4.5, where $d_0 = 3$, shows a very pronounced amplitude peak that rapidly decreases, signifying a strong initial impulse that swiftly dissipates due to internal damping. This behavior demonstrates the system's significant response to initial conditions, followed by a rapid return to a rest state and characteristics of effective control combined with significant boundary absorption.

On the other side, **Figures** 4.2 and **Fig.** 4.3, with d = 1, also shows an initial peak, but with a significantly lower amplitude than those observed in **Figures** 4.4 and **Fig.** 4.5. The decrease in amplitude is rapid, but less abrupt. These differences could indicate a less intense initial condition or a variation in damping. This suggests that the control remains effective, although it may be less intense or interact differently with the boundary conditions.

Now, let's analyze the energy associated to the problem (4.66)- (4.73) and (4.66)- (4.74).

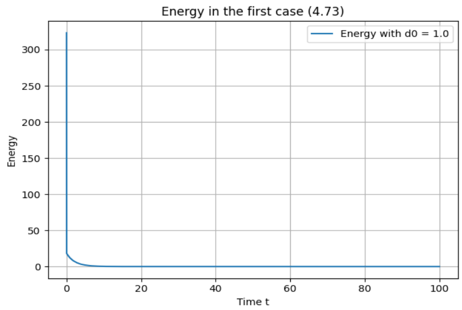

Figure 4.6: Energy's simulation $d_0 = 1$
Problem (4.66)-(4.73)

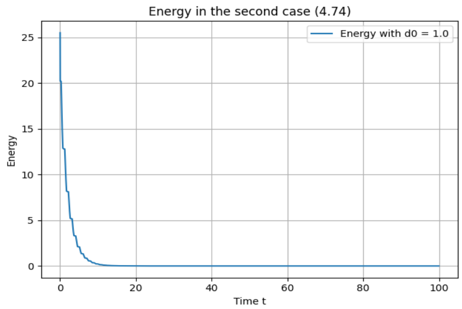

Figure 4.7: Energy's simulation $d_0 = 1$
Problem (4.66)-(4.74)

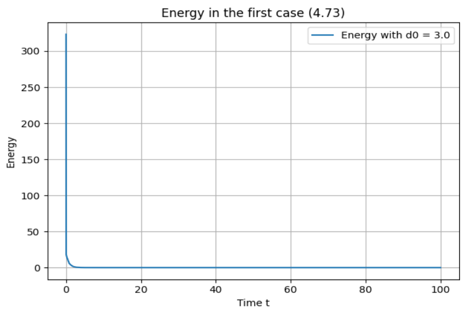

Figure 4.8: Energy's simulation $d_0 = 3$
Problem (4.66)-(4.73)

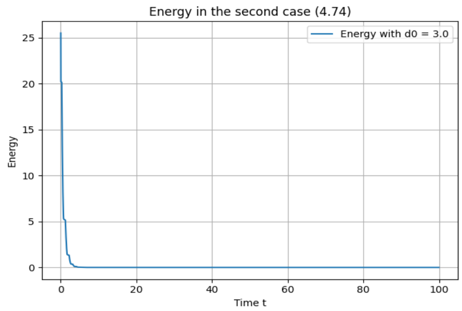

Figure 4.9: Energy's simulation $d_0 = 3$
Problem (4.66)-(4.74)

4.5.3 Analysis of energies

Fig. 4.6 and **Fig.** 4.7 illustrate a very rapid drop in energy at the start of the simulation, stabilizing almost immediately at a level close to zero. This evolution indicates a strong damping capacity, characterized by the rapid dissipation of initial energy. **Fig.** 4.8 and **Fig.** 4.9 also exhibit this rapid energy decline, but they stabilize the energy level at a slightly higher value than **Fig.** 4.6 and **Fig.** 4.7. This is attributable to the complexity of the initial conditions, which induce a more energetic initial state.

It thus appears that, regardless of the variations in initial conditions, the energy in the system ends up decreasing exponentially and reaches a remarkable precision of 10^{-10} from the time $T_N = T_0 + N\Delta t$ (see in the tables below).

Fig.4.6 and Fig.4.7: $d_0 = 1$				
$\vartheta_0(x)$	$\vartheta_1(x)$	Maximal value of E	Minimal index N at which $E < 10^{-10}$	Time T_N at N in seconds
$\sin\left(\frac{2\pi x}{L}\right)$	0	323.3	6698	66.98 s
$(x-1)e^{Lx}$	$\frac{(x-1)^2}{L}e^{Lx}$	25.5	6249	62.49 s

Fig.4.8 and Fig.4.9: $d_0 = 3$				
$\vartheta_0(x)$	$\vartheta_1(x)$	Maximal value of E	Minimal index N at which $E < 10^{-10}$	Time T_N at N in seconds
$\sin\left(\frac{2\pi x}{L}\right)$	0	323.3	2300	23s
$(x-1)e^{Lx}$	$\frac{(x-1)^2}{L}e^{Lx}$	25.5	2143	21.43 s

This observation underscores the effectiveness of internal damping and Wentzell boundary conditions in quickly regulating the energy in the system, ensuring efficient stabilization within very short time frames. It is important to note that all the graphs were plotted using $N_x = 49$ and a time interval $\Delta t = \frac{T}{N_t+1}$, where $N_t = 9999$. Therefore, the time T_N is calculated to be $\frac{N}{100}$ seconds.

4.6 Conclusion

The results from our simulations clearly demonstrate the significant influence of the damping coefficient on the rate at which energy dissipates in dynamic systems governed by a damped wave equation. Notably, a high level of damping leads to a more rapid stabilization of energy, which is advantageous in applications where it is crucial to quickly halt vibrations or oscillations. Conversely, low damping allows oscillations to persist longer, which can be beneficial in situations where the persistence of vibrations is required. Understanding these dynamics and being able to accurately predict the time needed to achieve energy stabilization is vital for the design and improvement of various devices and structures in mechanical and civil engineering. These insights provide a solid foundation for developing more effective control strategies in dynamic system management.

Regardless of the complexity of the initial conditions, our analysis reveals that the system loses energy exponentially, achieving a remarkable precision of 10^{-10} after a time interval defined by $T_N = T_0 + N\Delta t$. This observation validates the effectiveness of Wentzell boundary conditions and internal damping in stabilizing energy within the wave equation framework.

5

Exponential stability of a Locally Damped Wave with Significant Inertia at the Boundary: Method of Nakao

This chapter focuses on the statement of exponential decay result for an internal damped wave, by the use of Nakao's method. Contrary to the previous chapter where the boundary inertia is neglected, here the boundary Γ_1 has significant mass density which tends to resist to the interior dynamic.

5.1 Geometry of the medium

The geometry of the medium is the same as for the **Chapter** 4, that is:

$$\Omega = \{x \in \mathbb{R}^n : 0 < r_0 < \|x\|_2 < 1\} \subset \mathbb{R}^n, \quad \omega = \{x \in \mathbb{R}^n : r < \|x\|_2 < 1, r_0 < r < 1\},$$

and

$$\begin{aligned}
\Gamma_0 &= \{x \in \mathbb{R}^n : \|x\|_2 = r_0 > 0\}\\
\Gamma_1 &= \{x \in \mathbb{R}^n : \|x\|_2 = 1\}\\
\mathscr{I} &= \{x \in \mathbb{R}^n : \|x\|_2 = r, 0 < r_0 < r < 1\}
\end{aligned}$$

satisfying the conditions:

$$\begin{aligned}
&\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset\\
&\overline{\Gamma_i} \cap \overline{\mathscr{I}} = \emptyset, \forall i \in \{0, 1\}\\
&\partial\omega = \Gamma_1 \cup \mathscr{I} \, (\omega \subset \Omega \text{ is strict}).
\end{aligned}$$

This chapter merely focus on the published paper [14].

5.2 MODEL (5.1)

The model considered is same model studied in **Chapter** 3, that is:

$$\begin{cases} \vartheta_{tt}(x,t) - \Delta\vartheta(x,t) + d(x)\vartheta_t(x,t) = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \vartheta = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}_+, \\ \partial_\nu \vartheta + m(x)\vartheta_{tt} - \Delta_T \vartheta(x,t) = 0, & (x,t) \in \Gamma_1 \times \mathbb{R}_+, \\ \vartheta(x,0) = \vartheta_0(x), \quad \vartheta_t(x,0) = \vartheta_1(x), & x \in \Omega, \\ \vartheta_t(x,0) = \vartheta_2(x), & x \in \Gamma_1, \end{cases} \tag{5.1}$$

where $\vartheta = \vartheta(x,t)$ is the wave in $\overline{\Omega} \times \mathbb{R}_+$, and $(\vartheta_0, \vartheta_1, \vartheta_2)^\top$ represents the initial state of the system (5.1),

$d = d_0\chi_\omega$ maps Ω on $\mathbb{R}_+$ and allows a localized damping in Ω,

$m \in L^\infty(\Gamma_1)$ represents the mass density function on Γ_1 such that:

$$\exists\, m_0 > 0 \ : \ m(x) \geq m_0 > 0, \quad \forall x \in \Gamma_1.$$

Δ_T is the operator of Laplace-Beltrami [90] verifying:

$$\int_{\Gamma_1} v\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} \nabla_T u \nabla_T v \mathrm{d}\Gamma_1, \quad \forall u, v \in H^1(\Gamma_1). \tag{5.2}$$

In particular,

$$\int_{\Gamma_1} u\Delta_T u \mathrm{d}\Gamma_1 = -\int_{\Gamma_1} |\nabla_T u|^2 \mathrm{d}\Gamma_1, \quad \forall u \in H^1(\Gamma_1). \tag{5.3}$$

In addition, the eigenvalues of $-\Delta_T$ are non-negative in a riemanian manifold (see [80, p. 24]). Then it follows that:

$$\exists\, C_T > 0 \ : \ \|u\|^2_{L^2(\Gamma_1)} \leq C_T \|\nabla_T u\|^2_{L^2(\Gamma_1)}, \quad u \in H^1_{\Gamma_0}(\Omega) \tag{5.4}$$

where

$$H^1_{\Gamma_0}(\Omega) = \left\{ z \in H^1(\Omega) \ : \ z_{/\Gamma_0} = 0 \right\}.$$

equipped by the norm:

$$\|.\|^2_{H^1_{\Gamma_0}(\Omega)} = \|\nabla .\|^2_{L^2(\Omega)} + \|\nabla_T .\|^2_{L^2(\Gamma_1)}.$$

The minimal state space $\mathscr{H}$ is defined as:

$$\mathscr{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1).$$

with the inner norm:

$$\|(u,v,w)^\top\|^2_{\mathscr{H}} = \|u\|^2_{H^1_{\Gamma_0}(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|m^{\frac{1}{2}}(x)w\|^2_{L^2(\Gamma_1)}, \quad \forall (u,v,w)^\top \in \mathscr{H}$$

For readers convenience, we repeat the regularity theorem and the stability result of the system (5.1).

Theorem 5.2.1. *(Regularity) If $U_0 \in \mathscr{H}$, then system* (5.1) *admits a unique weak solution*

$$U \in C([0,+\infty), \mathscr{H}).$$

Moreover, if $U_0 \in D(A)$, then system (5.1) *admits a unique strong solution*

$$U \in C([0,+\infty), D(A)) \cap C^1([0,+\infty), \mathscr{H}).$$

Proof. See **Chapter** 3. □

Proposition 5.2.2. *(Stability) The system* (5.1) *is well-posed in the state space $\mathscr{H}$. Moreover, the energy ξ defined as:*

$$\xi(t) = \frac{1}{2}\left[\int_\Omega |\nabla\vartheta|^2 \mathrm{d}x + \int_{\Gamma_1} |\nabla_T\vartheta|^2 \mathrm{d}\Gamma_1 + \int_\Omega |\vartheta_t|^2 \mathrm{d}x + \int_{\Gamma_1} m(x)|\vartheta_t|^2 \mathrm{d}\Gamma_1\right], \tag{5.5}$$

is dissipative such that:

$$\frac{d\xi(t)}{dt} = -\int_\Omega d(x)|\vartheta_t|^2 \mathrm{d}x. \tag{5.6}$$

Proof. See **Chapter** 3. □

Now, let's state the exponential stability of the system (5.1) by the Nakao's approach.

5.3 Exponential stability & Nakao method

Here we need some initial results, which are useful in Nakao's approach.

5.3.1 Some results

Lemma 5.3.1. *Let $(\vartheta, \vartheta_t, \vartheta_{t/_{\Gamma_1}})^\top \in \mathscr{H}$ be a solution of the system* (5.1)*. Suppose that $\vartheta_t \neq 0$ almost everywhere in $\omega \times (T_0, T_1)$, for some $0 \leq T_0 < T_1$. Then there exists a constant $\tau_Q > 0$, depending on $Q \in \{\Omega, \Gamma_1\}$, such that*

$$\int_{T_1}^{T_0} \int_Q |\vartheta_t(x,s)|^2 \,\mathrm{d}Q\,\mathrm{d}s \leq \tau_Q \int_{T_0}^{T_1} \int_\omega |\vartheta_t(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s, \tag{5.7}$$

with

$$\tau_Q = \begin{cases} \tau_\omega \geq 1, & if\, Q = \Omega, \\ \tau(\Gamma_1) > 0, & if\, Q = \Gamma_1. \end{cases}$$

Proof. This can be assumed according to the observability's result of **Chapter 3**. The proof is complete. □

Lemma 5.3.2. *Let* $(\vartheta, \vartheta_t, \vartheta_{t|_{\Gamma_1}})^\top \in \mathscr{H}$ *be a solution of the system* (5.1). *Then, for all instant* $t \geq 0$ *there exists two instants* $t_1 \in [t, t + \frac{1}{4}]$ *and* $t_2 \in [t + \frac{3}{4}, t + 1]$ *such that:*

$$\int_\omega |\vartheta_t(t_i)|^2 \mathrm{d}x \leq \frac{4}{d_0} \int_t^{t+1} \int_\Omega d(x)|\vartheta_t|^2 \mathrm{d}x, \quad \forall i \in \{1, 2\}. \tag{5.8}$$

where $\vartheta_t(t_i) = \vartheta_t(x, t_i)$.

Proof. Define

$$I_1 = [t, t + \frac{1}{4}], \quad I_1 = [t + \frac{3}{4}, t + 1].$$

and

$$F(t) := d_0 \|\vartheta_t(t)\|^2_{L^2(\omega)}, \quad t \geq 0.$$

It is obvious that $t \mapsto F(t)$ is non-negative and continuous over each interval $[t, t + 1], t \geq 0$. Consequently, by Weiertrass's theorem on continuous function over a compact domain, F is bounded and attains its extremum values. In this sense, let's choose $t_i \in I_i$, $i \in \{1, 2\}$ such that:

$$F(t_i) = \underset{s \in I_i}{min} F(s), \quad i \in \{1, 2\}.$$

Then the following inequality is obvious:

$$F(t_i) \leq F(s), \forall s \in I_i, \tag{5.9}$$

Integrating over I_i, it follows for all $i \in \{1, 2\}$:

$$\int_{I_i} F(t_i) \mathrm{d}s = F(t_i)\,\mathrm{meas}(I_i) \leq \int_{I_i} F(s) \mathrm{d}s.$$

where meas($\mathrm{I_i}$) is the standard measure of segment.

It is obvious that meas($\mathrm{I_1}$) = meas(I_2) = $\frac{1}{4}$. Then, it follows that:

$$F(t_i) \leq 4 \int_{I_i} F(s) \mathrm{d}s, \quad \forall i \in \{1, 2\}.$$

Since $I_i \subset [t, t + 1]$, then

$$F(t_i) \leq 4 \int_t^{t+1} F(s) \mathrm{d}s, \quad \forall i \in \{1, 2\}.$$

Replacing F by its value, we get the desired resuly.

The proof is complete. □

5.3.2 Exponential Decay

Lemma 5.3.3. *Let $\xi(t)$ be a bounded, positive function defined on $[0,+\infty)$. Assume there exists a constant $C_0 > 1$ such that*

$$\sup_{s\in[t,t+1]} \xi(s) \leq C_0 \left(\xi(t) - \xi(t+1)\right), \quad \forall t \geq 0. \tag{5.10}$$

Then, there exist positive constants M and α such that

$$\xi(t) \leq Me^{-\alpha t}, \quad \forall t \geq 0.$$

Proof. See [12, Lemma 31, p. 406]. See also [75, Lemma 3, p. 339]. □

Now, we are in position to state the exponential stability result.

Theorem 5.3.4. *Let $(\vartheta, \vartheta_t, \vartheta_{t/_{\Gamma_1}})^\top \in \mathscr{H}$ be a solution of the system* (5.1). *If $\vartheta_{t/_\omega} \neq 0$, then there exist constants $M > 1$ and $\alpha > 0$ such that the energy $\xi(t)$ given by* (4.28) *satisfies the following inequality:*

$$\xi(t) \leq Me^{-\alpha t}\xi(0), \quad \forall t \geq 0, \tag{5.11}$$

whend_0 is under a certain condition.

Proof. First, observe that if $\vartheta_t = 0$ over ω, the energy ξ satisfies $\xi'(t) = 0$ thanks to (5.6). Then ξ remains constant ($\xi(t) = \xi(0)$ for all $t \geq 0$), and hence ξ is not exponentially stable unless $\xi(t) = \xi(0) = 0$. Therefore, we assume that:

$$\vartheta_{t/_\omega} \neq 0, \quad R = \sup_{x\in\Gamma_1} |m(x)|.$$

Next, integrating (5.6) over $[t, t+1]$, we obtain:

$$\xi(t) - \xi(t+1) = \int_t^{t+1} \int_\Omega d(x)|\vartheta_t(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.12}$$

Let $K : \mathbb{R}_+ \to \mathbb{R}_+$ be such that

$$K^2(t) = \xi(t) - \xi(t+1), \quad \forall t \in [0, \infty). \tag{5.13}$$

Thus, from **Lemma** 5.3.2, there exists $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t+1]$ such that:

$$\int_\omega |\vartheta_t(x,t_i)|^2 \,\mathrm{d}x \leq \frac{4}{d_0} K^2(t), \quad \forall i \in \{1, 2\}. \tag{5.14}$$

From now on, the methodology follows the procedures outlined in [12, 75, 78].

Multiplying $(5.1)_1$ by ϑ and integrating by parts over Ω, we obtain:

$$\int_\Omega \vartheta_{tt}\vartheta \,\mathrm{d}x - \int_\Omega \vartheta\Delta\vartheta \,\mathrm{d}x + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0. \tag{5.15}$$

Knowing that $\vartheta_{tt}\vartheta = \frac{d}{dt}(\vartheta_t\vartheta) - |\vartheta_t|^2$, we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \vartheta_t\vartheta \,\mathrm{d}x\right) - \int_\Omega |\vartheta_t|^2 \,\mathrm{d}x\\ &- \int_\Omega \vartheta\Delta\vartheta \,\mathrm{d}x + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0.\end{aligned}$$

Then, using Green's formula and the fact that $\vartheta_{|\Gamma_0} = 0$, we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \vartheta_t\vartheta \,\mathrm{d}x\right) - \int_\Omega |\vartheta_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x\\ &\qquad - \int_{\Gamma_1} \vartheta\partial_\nu\vartheta \,\mathrm{d}\Gamma_1 + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0.\end{aligned} \tag{5.16}$$

Since $\partial_\nu\vartheta = \Delta_T\vartheta - m\vartheta_{tt}$ on Γ_1, and using the property (5.3), we arrive at:

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \vartheta_t\vartheta \,\mathrm{d}x\right) + \frac{d}{dt}\left(\int_{\Gamma_1} m\vartheta_t\vartheta \,\mathrm{d}\Gamma_1\right)\\ &\qquad + \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x + \int_{\Gamma_1} |\nabla_T\vartheta|^2 \,\mathrm{d}\Gamma_1\\ &\qquad - \int_\Omega |\vartheta_t|^2 \,\mathrm{d}x - \int_{\Gamma_1} m|\vartheta_t|^2 \,\mathrm{d}\Gamma_1 + \int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x = 0.\end{aligned} \tag{5.17}$$

Next, integrating (5.17) over $[t_1, t_2]$, we get:

$$\int_{t_1}^{t_2} \left(\|\nabla\vartheta\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)}\right) \mathrm{d}s = I_1 - I_2 + \int_{t_1}^{t_2} \left(\|\vartheta_t\|^2_{L^2(\Omega)} + \|\sqrt{m}\vartheta_t\|^2_{L^2(\Gamma_1)}\right) \mathrm{d}s - J. \tag{5.18}$$

where

$$\begin{aligned}I_i &= \int_\Omega \vartheta_t(x,t_i)\vartheta(x,t_i) \,\mathrm{d}x + \int_{\Gamma_1} m\vartheta_t(x,t_i)\vartheta(x,t_i) \,\mathrm{d}\Gamma_1, \quad \forall i \in \{1,2\},\\ J &= \int_{t_1}^{t_2}\int_\Omega d(x)\vartheta_t\vartheta \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Using Cauchy-Schwarz's inequality and (5.7), we get for all $i \in \{1,2\}$:

$$\begin{aligned}|I_i| &\le \|\vartheta_t(t_i)\|_{L^2(\Omega)} \|\vartheta(t_i)\|_{L^2(\Omega)} + \|m\vartheta_t(t_i)\|_{L^2(\Gamma_1)} \|\vartheta(t_i)\|_{L^2(\Gamma_1)}\\ &\le \sqrt{\tau_\omega}\|\vartheta_t(t_i)\|_{L^2(\omega)} \|\vartheta(t_i)\|_{L^2(\Omega)} + R^2 \|\vartheta_t(t_i)\|_{L^2(\Gamma_1)} \|\vartheta(t_i)\|_{L^2(\Gamma_1)}\\ &\le \sqrt{\tau_\omega}\|\vartheta_t(t_i)\|_{L^2(\omega)} \|\vartheta(t_i)\|_{L^2(\Omega)} + R^2\sqrt{C_T}\|\vartheta_t(t_i)\|_{L^2(\Gamma_1)} \|\nabla_T\vartheta(t_i)\|_{L^2(\Gamma_1)}.\end{aligned} \tag{5.19}$$

Using (5.14) and Poincaré's inequality along with (5.4), we obtain:

$$\begin{aligned}|I_i| &\leq 2c_p\sqrt{\frac{\tau_\omega}{d_0}}K(t)\,\|\nabla\vartheta(t_i)\|_{L^2(\Omega)} + R^2\,\|\vartheta_t(t_i)\|_{L^2(\Gamma_1)}\,\|\vartheta(t_i)\|_{L^2(\Gamma_1)} \\ &\leq 2c_p\sqrt{\frac{\tau_\omega}{d_0}}K(t)\,\|\nabla\vartheta(t_i)\|_{L^2(\Omega)} + R^2\sqrt{C_T}\,\|\vartheta_t(t_i)\|_{L^2(\Gamma_1)}\,\|\nabla_T\vartheta(t_i)\|_{L^2(\Gamma_1)}\,.\end{aligned} \tag{5.20}$$

Applying the necessary condition (5.7), we get the bound:

$$|I_i| \leq C_0K(t)\left(\|\nabla\vartheta(t_i)\|_{L^2(\Omega)} + \|\nabla_T\vartheta(t_i)\|_{L^2(\Gamma_1)}\right), \tag{5.21}$$

where $C_0 = \max\left(2c_p\sqrt{\frac{\tau_\omega}{d_0}}; 2R^2\sqrt{\frac{\tau(\Gamma_1)C_T}{d_0}}\right)$.

Thanks to Cauchy-Schwarz's inequality, we have:

$$|J| \leq d_0\int_{t_1}^{t_2}\|\vartheta_t\|_{L^2(\omega)}\,\|\vartheta\|_{L^2(\omega)}\,\mathrm{d}s. \tag{5.22}$$

Using the fact that $ab \leq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}$ and Poincaré's inequality, we get:

$$\begin{aligned}|J| &\leq \frac{d_0}{2}\left(\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\omega)}\,\mathrm{d}s + \int_{t_1}^{t_2}\|\vartheta\|^2_{L^2(\omega)}\,\mathrm{d}s\right) \\ &\leq \frac{d_0}{2}\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\omega)}\,\mathrm{d}s + \frac{d_0c_p}{2}\int_{t_1}^{t_2}\|\nabla\vartheta\|^2_{L^2(\omega)}\,\mathrm{d}s \\ &\leq \frac{d_0}{2}\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\omega)}\,\mathrm{d}s + \frac{d_0c_p}{2}\int_{t_1}^{t_2}\|\nabla\vartheta\|^2_{L^2(\Omega)}\,\mathrm{d}s,\end{aligned} \tag{5.23}$$

where $c_p > 0$ is Poincaré's constant.

Using the definition of K from (5.12)- (5.13), it follows that

$$|J| \leq \frac{1}{2}K^2(t) + \frac{d_0c_p}{2}\int_{t_1}^{t_2}\|\nabla\vartheta\|^2_{L^2(\Omega)}\,\mathrm{d}s. \tag{5.24}$$

Combining (5.24) and (5.21) in (5.17), we obtain:

$$\begin{aligned}\left(1-\frac{c_pd_0}{2}\right)\int_{t_1}^{t_2}\|\nabla\vartheta\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_{t_1}^{t_2}\|\nabla_T\vartheta\|^2_{L^2(\Gamma_1)}\,\mathrm{d}s \leq C_0K(t)S + \frac{1}{2}K^2(t)& \\ + \int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\Omega)}\,\mathrm{d}s& \\ + \int_{t_1}^{t_2}\left\|m^{\frac{1}{2}}\vartheta_t\right\|^2_{L^2(\Gamma_1)}\,\mathrm{d}s,&\end{aligned} \tag{5.25}$$

where

$$S=\sum_{i=1}^{2}\left(\|\nabla\vartheta(t_i)\|_{L^2(\Omega)}+\|\nabla_T\vartheta(t_i)\|_{L^2(\Gamma_1)}\right).$$

Taking $d_0<\frac{2}{c_p}$ and using (5.7) in the second member of (5.25), we get:

$$\int_{t_1}^{t_2}\|\vartheta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)}\,\mathrm{d}s\le\frac{2}{2-c_pd_0}\left[C_0K(t)S+\frac{1}{2}K^2(t)+(\tau_\omega+R\tau(\Gamma_1))\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\omega)}\,\mathrm{d}s\right].\tag{5.26}$$

Using the definition of K from (5.12)- (5.13), it follows that:

$$\int_{t_1}^{t_2}\|\vartheta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)}\,\mathrm{d}s\le\mu_0\left(K(t)S+K^2(t)\right),\tag{5.27}$$

where

$$\mu_0=\frac{2}{2-c_pd_0}\max\left(C_0,\frac{1}{2}+\frac{\tau_\omega+R\tau(\Gamma_1)}{d_0}\right).$$

Adding $\int_{t_1}^{t_2}\left\|m^{\frac{1}{2}}\vartheta_t\right\|^2_{L^2(\Gamma_1)}\,\mathrm{d}s$ to both sides of (5.27) and then majoring the right-hand side by using (5.7), we get:

$$\begin{aligned}\int_{t_1}^{t_2}\left(\|\vartheta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)}+\left\|m^{\frac{1}{2}}\vartheta_t\right\|^2_{L^2(\Gamma_1)}\right)\mathrm{d}s&\le\mu_0\left(K(t)S+K^2(t)\right)+\int_{t_1}^{t_2}\left\|m^{\frac{1}{2}}\vartheta_t\right\|^2_{L^2(\Gamma_1)}\,\mathrm{d}s\\&\le\mu_0\left(K(t)S+K^2(t)\right)+R\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\Gamma_1)}\,\mathrm{d}s\\&\le\mu_0\left(K(t)S+K^2(t)\right)+R\tau(\Gamma_1)\int_{t_1}^{t_2}\|\vartheta_t\|^2_{L^2(\omega)}\,\mathrm{d}s\\&\le\mu_0\left(K(t)S+K^2(t)\right)+\frac{R\tau(\Gamma_1)}{d_0}K^2(t)\\&\le\mu_1\left(K(t)S+K^2(t)\right),\end{aligned}\tag{5.28}$$

where $\mu_1=\max\left(\mu_0,\frac{R}{d_0}\tau(\Gamma_1)\right)$.

It is obvious that:

$$|S|\le 2\sup_{s\in[t;t+1]}\left(\|\nabla\vartheta(s)\|_{L^2(\Omega)}+\|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)}\right).$$

Then

$$\int_{t_1}^{t_2}\left(\|\vartheta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)}+\left\|m^{\frac{1}{2}}\vartheta_t\right\|^2_{L^2(\Gamma_1)}\right)\mathrm{d}s\le\mu_1\left[2K(t)\sup_{s\in[t;t+1]}\left(\|\nabla\vartheta(s)\|_{L^2(\Omega)}+\|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)}\right)+K^2(t)\right]\tag{5.29}$$

Summing (5.7) to (5.29) side by side, we get

$$\int_{t_1}^{t_2} \left(\|\vartheta\|^2_{\mathscr{H}^1_{\Gamma_0}(\Omega)} + \left\| m^{\frac{1}{2}} \vartheta_t \right\|^2_{L^2(\Gamma_1)} + \|\vartheta_t\|^2_{L^2(\Omega)} \right) \mathrm{d}s \leq \tilde{K}^2(t) + \frac{\tau_\omega}{d_0} K^2(t) \tag{5.30}$$

with

$$\tilde{K}^2(t) = \mu_1 \left[2K(t) \sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} + \|\nabla_T \vartheta(s)\|_{L^2(\Gamma_1)} \right) + K^2(t) \right].$$

It follows that there exists $t_0 \in [t_1, t_2]$ such that

$$\|\nabla\vartheta(t_0)\|^2_{L^2(\Omega)} + \|\nabla_T\vartheta(t_0)\|^2_{L^2(\Gamma_1)} + \left\| \sqrt{m}\vartheta_t(t_0) \right\|^2_{L^2(\Gamma_1)} + \|\vartheta_t(t_0)\|^2_{L^2(\Omega)} \leq 2\left(\tilde{K}^2(t) + \frac{\tau_\omega}{d_0} K^2(t) \right) \tag{5.31}$$

which is equivalent to

$$\xi(t_0) \leq 2\left(\tilde{K}^2(t) + \frac{\tau_\omega}{d_0} K^2(t) \right) \tag{5.32}$$

Now, integrating by parts over $[t, t_0]$ the equation (5.6), we get

$$\xi(t) = \xi(t_0) + \int_t^{t_0} \int_\Omega d(x) |\vartheta_t|^2 \mathrm{d}x\mathrm{d}s \tag{5.33}$$

From (5.13) and (5.32), we obtain

$$\begin{aligned} \underset{s\in[t;t+1]}{sup}\ \xi(s) &\leq \left(1 + 2\frac{\tau_\omega}{d_0}\right) K^2(t) + 2\mu_1 \left[K(t) \sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} + \|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)} \right) + K^2(t) \right] \\ &\leq \left[1 + 2\left(\frac{\tau_\omega}{d_0} + \mu_1 \right) \right] K^2(t) + 2\mu_1 K(t) \sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} + \|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)} \right) \end{aligned}$$

Since

$$\sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} + \|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)} \right) \leq \sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} \right) + \sup_{s\in[t;t+1]} \left(\|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)} \right)$$

and

$$\left\| \begin{aligned} &\sup_{s\in[t;t+1]} \left(\|\nabla\vartheta(s)\|_{L^2(\Omega)} \right) \leq \sup_{s\in[t;t+1]} \left(\sqrt{2\xi(s)} \right) \\ &\sup_{s\in[t;t+1]} \left(\|\nabla_T\vartheta(s)\|_{L^2(\Gamma_1)} \right) \leq \sup_{s\in[t;t+1]} \left(\sqrt{2\xi(s)} \right) \end{aligned} \right.$$

Then using the inequality $ab \leq \frac{a^2+b^2}{2}$, $a, b \in \mathbb{R}$, we get

$$\begin{aligned} \underset{s\in[t;t+1]}{sup}\ \xi(s) &\leq \left[1 + 2\left(\frac{\tau_\omega}{d_0} + \mu_1 \right) \right] K^2(t) + 4\mu_1 K(t) \underset{s\in[t;t+1]}{sup} \left(\sqrt{2\xi(s)} \right) \\ &\leq \left[1 + 2\left(\frac{\tau_\omega}{d_0} + \mu_1 \right) + 16\mu_1^2 \right] K^2(t) + \frac{1}{2} \underset{s\in[t;t+1]}{sup} \left(\xi(s) \right) \end{aligned} \tag{5.34}$$

and then

$$\frac{1}{2}\sup_{s\in[t;t+1]}(\xi(s)) \le \left[1 + 2\left(\frac{\tau_\omega}{d_0} + \mu_1\right) + 16\mu_1^2\right]K^2(t)$$

hence

$$\sup_{s\in[t;t+1]}(\xi(s)) \le C\left[\xi(t) - \xi(t+1)\right],$$

where $C = 2\left[1 + 2\left(\frac{\tau_\omega}{d_0} + \mu_1\right) + 16\mu_1^2\right] > 1$.

According to **Lemma** 5.3.3 and the result (4.53) from **Section** 4.4.3 of **Chapter** 4, this result thus validates the exponential stability of energy ξ. Moreover,

$$\xi(t) \le \frac{C}{C-1} e^{log\left(1-\frac{1}{C}\right)t}\xi(0), \quad \forall t > 1. \tag{5.35}$$

The Proof is complete. □

Remark 5.3.5. *These findings clarify the necessary conditions to be set for the Nakao method to be used in order to benefit from the exponential stability of such a problem with an internal and dynamic controller. The Nakao method becomes complex, if not almost impossible, when $d_0 \ge \frac{2}{c_p}$, as can be seen.*

Due to the fact that exponential stability is correlated with the constants τ_ω and $\tau(\Gamma_1)$, it is also correlated with the parameters d_0 and m, respectively, which enable much stronger control over $\overline{\Omega}$.

Now, let's see the influence of $d_0, R, c_p, \tau_\omega, \tau(\Gamma_1)$ on the exponential stability.

5.3.3 Impact of parameters

Here are the conditions which lead to the lost of exponential decay rate as in **Section** 4.4.3 of **Chapter** 4. For this aim, consider the parameters outlines from Nakao's method as $C_0 = C_0(d_0, R, \tau_Q)$, $\mu_0 = \mu_0(d_0, R, \tau_Q)$, and $\mu_1 = \mu_1(d_0, R, \tau_Q)$ with τ_Q defined in **Lemma** 5.3.1.

Remark 5.3.6. *(Changes in C_0)*
The parameter $C_0 \to +\infty$ if:

i. $R \to +\infty$ *or* $d_0 \to 0^+$. *In particular,* $d_0 \to 0^+$ *if* $c_p \to +\infty$.

ii. $\tau_Q \to +\infty$.

Remark 5.3.7. *(Changes in μ_0)*
The parameter $\mu_0 \to +\infty$ if:

i. $C_0 \to +\infty$.

ii. $d_0 \underset{d_0<\frac{2}{c_p}}{\longrightarrow} \frac{2}{c_p}$. *In particular, d_0 may tend to ∞ if $c_p \to 0^+$.*

Remark 5.3.8. *(Changes in μ_1 and C)*
The following assertions are obvious:

i. if $\mu_0 \to +\infty$ then $\mu_1 \to +\infty$.

ii. if $\mu_1 \to +\infty$ then $C \to +\infty$.

The following result yield from the last three remarks.

Proposition 5.3.9. *Let $(\vartheta, \vartheta_t, \vartheta_{t|_{\Gamma_1}})^\top \in \mathscr{H}$ be a solution of system* (5.1). *Then the exponential decay rate established in* (5.35) *gets lost when:*

i. $R \to +\infty$ or $d_0 \to 0^+$. In particular, $d_0 \to 0^+$ if $c_p \to +\infty$.

ii. $d_0 \underset{d_0<\frac{2}{c_p}}{\longrightarrow} \frac{2}{c_p}$. *In particular, d_0 may tend to ∞ if $c_p \to 0^+$.*

iii. $\tau_Q \to +\infty$.

Remark 5.3.10. *Since the decay rate gets lost when d_0 tends to 0^+ or $\frac{2}{c_p}$ then for a given $R \geq m_0 > 0$, there exists a coefficient of dissipation $a_0 < \frac{2}{c_p}$ such that:*

- *the more d_0 increases in $]0, a_0[$, the more the exponential decay rate hold.*
- *the more d_0 increases in $]a_0, \frac{2}{c_p}[$, the more the exponential decay rate gets lost.*

Moreover, $R \mapsto C(R)$ is a monotone increasing function in $[m_0, +\infty[$. Thus, the exponential decay rate gets lost when R increases in $[m_0, +\infty[$.

5.4 Numerical Analysis

In this section, we demonstrate the exponential stability of the system (5.1) in one spatial dimension using the finite difference method. To properly handle the boundary condition $(5.1)_2$, which simultaneously involves normal and tangential derivatives, we begin by analyzing the system (5.1) in two dimensions. Then, we employ a partial Fourier series expansion to reduce the system to an equivalent one-dimensional form.

Consider the two-dimensional system:

$$\begin{cases} \vartheta_{tt}((x,y),t) - \vartheta_{xx}((x,y),t) - \vartheta_{yy}((x,y),t) + d(x,y)\vartheta_t((x,y),t) = 0, & ((x,y),t) \in \Omega \times \mathbb{R}_+, \\ m(x,y)\vartheta_{tt}((x,y),t) - \Delta_T \vartheta((x,y),t) + \partial_\nu \vartheta((x,y),t) = 0, & ((x,y),t) \in \Gamma_1 \times \mathbb{R}_+, \\ \vartheta((x,y),t) = 0, & ((x,y),t) \in \Gamma_0 \times \mathbb{R}_+, \\ \vartheta((x,y),0) = \vartheta_0(x,y), \quad \vartheta_t((x,y),0) = \vartheta_1(x,y), & (x,y) \in \Omega. \end{cases} \tag{5.36}$$

where Ω is an open subset of $\mathbb{R}^2$:

$$\Omega = \{u = (x,y) \in \mathbb{R}^2 \,:\, 0 < r_0 < |u| = x^2 + y^2 < 1\} \tag{5.37}$$

with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that

$$\Gamma_0 = \{u \in \mathbb{R}^2 |\, |u| = r_0\}, \quad \Gamma_1 = \{u \in \mathbb{R}^2 |\, |u| = 1\} \tag{5.38}$$

We assume the following partial Fourier series representation:

$$\vartheta((x,y),t) = \sum_{k\geq 0} \vartheta^{(k)}(x,t)\cos(Ly), \quad \forall\, ((x,y),t) \in \Omega \times \mathbb{R}_+, \tag{5.39}$$

where $\vartheta^{(k)}$ are the Fourier coefficients and $L > 0$ may depends on k.

Let's inject (5.39) in (5.36), it is clear that, on Ω we obtain

$$\begin{aligned} &\sum_{k\geq 0} \vartheta_{tt}^{(k)}(x,t)cos(Ly) - \sum_{k\geq 0} \vartheta_{xx}^{(k)}(x,t)cos(Ly) \\ &+ L^2 \sum_{k\geq 0} \vartheta^{(k)}(x,t)cos(Ly) + d(x,y) \sum_{k\geq 0} \vartheta_t^{(k)}(x,t)cos(Ly) = 0 \end{aligned} \tag{5.40}$$

which implies

$$\vartheta_{tt}^{(k)}(x,t) - \vartheta_{xx}^{(k)}(x,t) + L^2 \vartheta^{(k)}(x,t) + d(x,y)\vartheta_t^{(k)}(x,t) = 0. \tag{5.41}$$

Since $\partial_\nu f = \nabla f.\nu$, $f \in C^1$ and (.) is the standard inner product of $\mathbb{R}^2$, then

$$\partial_\nu \vartheta((x,y),t) = x\vartheta_x((x,y),t) + y\vartheta_y((x,y),t), \quad \forall (x,y) \in \Gamma_1,\ t \geq 0 \tag{5.42}$$

Then using (5.39), we get

$$\partial_\nu \vartheta((x,y),t) = \sum_{k\geq 0} \left(x\vartheta_x^{(k)}(x,t)cos(Ly) - Ly\vartheta^{(k)}(x,t)sin(Ly) \right). \tag{5.43}$$

We also have

$$\begin{aligned}\Delta_T\vartheta((x,y),t) &= \frac{\partial^2}{\partial y^2}\left(\sum_{k\geq 0}\vartheta^{(k)}(x,t)cos(Ly)\right)\\ &= -L^2\sum_{k\geq 0}\vartheta^{(k)}(x,t)cos(Ly)\end{aligned} \tag{5.44}$$

Thus, the boundary condition $m(x,y)\vartheta_{tt} - \Delta_T\vartheta + \partial_\nu\vartheta = 0$ over Γ_1, is

$$\sum_{k\geq 0}\left[\left(m(x,y)\vartheta_{tt}^{(k)}(x,t) + x\vartheta_x^{(k)}(x,t) + L^2\vartheta^{(k)}(x,t)\right)cos(Ly) - Ly\vartheta^{(k)}(x,t)sin(Ly)\right] = 0 \tag{5.45}$$

To reduce the problem to one dimension while preserving the parameter m, it suffices to choose $y = 0$ or $y = \frac{k\pi}{L}$, with $k \in \mathbb{N}^*$. We therefore chose $y = 0$ to have a one dimensional case , and the only possible values for x are 1 and -1 on Γ_1 since $x^2 + y^2 = 1$. To simplify the numerical study, we limit ourselves to $x = 1$ only. Then (5.45) becomes

$$\sum_{k\geq 0}\left[\left(m(1,0)\vartheta_{tt}^{(k)}(1,t) + \vartheta_x^{(k)}(1,t) + L^2\vartheta^{(k)}(1,t)\right)\right] = 0, \tag{5.46}$$

Which implies

$$m(1)\vartheta_{tt}^{(k)}(1,t) + \vartheta_x^{(k)}(1,t) + L^2\vartheta^{(k)}(1,t) = 0 \tag{5.47}$$

where $m(1) = m(1,0)$.

Moreover, the Dirichlet condition imposes that $\vartheta^{(k)}(r_0,t) = 0$.
This last condition combined with (5.41) and (5.47) allows us to deduce the following one-dimensional problem:

$$\begin{cases}\vartheta_{tt}^{(k)}(x,t) - \vartheta_{xx}^{(k)}(x,t) + L^2\vartheta^{(k)}(x,t) + d(x)\vartheta_t^{(k)}(x,t) = 0, & (x,t) \in \Omega_x \times \mathbb{R}_+,\\ m(1)\vartheta_{tt}^{(k)}(1,t) + \vartheta_x^{(k)}(1,t) + L^2\vartheta^{(k)}(1,t) = 0, & t \in \mathbb{R}_+,\\ \vartheta^{(k)}(r_0,t) = 0, & t \in \mathbb{R}_+,\\ \vartheta^{(k)}(x,0) = \vartheta_0^{(k)}(x), \quad \vartheta_t^{(k)}(x,0) = \vartheta_1^{(k)}(x), & x \in \Omega_x,\end{cases} \tag{5.48}$$

where $\Omega_x = (r_0, 1)$, and the damping region is defined as $\omega_x = (r, 1)$ with $r_0 < r < 1$.

Remark 5.4.1. *In the case where $y = y_0 \neq 0$ is chosen, L is replaced by kL in equation* (5.39)*, and consequently $L = \frac{\pi}{y_0}$. In this case, L represents fundamental spatial frequency in y-direction and $L \geq \pi$ since $y_0 \in (0,1)$.*

For simplicity, we will drop the index k.

We discretize the domain Ω_x into $N_x + 1 \in \mathbb{N}$ spatial sub-intervals, and the time interval $[0, T]$ into $N_t + 1 \in \mathbb{N}$ steps as follows:

$$\begin{cases} x_0 = r_0, \\ \Delta x = \dfrac{1 - r_0}{N_x + 1}, \\ x_\imath = x_0 + \imath \Delta x, \quad \forall \imath \in \{0, 1, \dots, N_x\}, \\ x_{N_x+1} = 1, \end{cases} \quad \text{and} \quad \begin{cases} T_0 = 0, \\ \Delta t = \dfrac{T}{N_t + 1}, \\ T_\jmath = T_0 + \jmath \Delta t, \quad \forall \jmath \in \{0, 1, \dots, N_t\}, \\ T_{N_t+1} = T. \end{cases} \tag{5.49}$$

Let $\vartheta(x_\imath, t_\jmath) = \vartheta_\imath^\jmath$ and $d(x_\imath) = d_\imath$.

By approximating the derivatives in (5.48) using finite differences, we obtain the following numerical scheme:

$$\begin{cases} \dfrac{\vartheta_\imath^{\jmath+1} - 2\vartheta_\imath^\jmath + \vartheta_\imath^{\jmath-1}}{(\Delta t)^2} - \dfrac{\vartheta_{\imath+1}^\jmath - 2\vartheta_\imath^\jmath + \vartheta_{\imath-1}^\jmath}{(\Delta x)^2} + L^2 \vartheta_\imath^\jmath + d_\imath \dfrac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1}}{2\Delta t} = 0, \\ m(1) \dfrac{\vartheta_{N_x+1}^{\jmath+1} - 2\vartheta_{N_x+1}^\jmath + \vartheta_{N_x+1}^{\jmath-1}}{(\Delta t)^2} + L^2 \vartheta_{N_x+1}^\jmath + \dfrac{1}{\Delta x}\left(\vartheta_{N_x+1}^\jmath - \vartheta_{N_x}^\jmath\right) = 0, \\ \vartheta_0^\jmath = 0, \\ \vartheta_\imath^0 = \vartheta_0(x_\imath), \quad \vartheta_\imath^1 = \vartheta_\imath^0 + \Delta t \, \vartheta_1(x_\imath). \end{cases} \tag{5.50}$$

The total energy associated with the discretized system (5.50) is given by:

$$\begin{aligned} \xi_\jmath = {} & \frac{m(1)}{2} \left| \frac{\vartheta_{N_x+1}^{\jmath+1} - \vartheta_{N_x+1}^\jmath}{\Delta t} \right|^2 + \frac{L^2}{2} \vartheta_{N_x+1}^{\jmath+1} \vartheta_{N_x+1}^\jmath + \frac{L^2}{2} \Delta x \sum_{\imath=0}^{N_x} \vartheta_\imath^{\jmath+1} \vartheta_\imath^\jmath \\ & + \frac{\Delta x}{2} \sum_{\imath=0}^{N_x} \left(\frac{\vartheta_{\imath+1}^{\jmath+1} - \vartheta_\imath^{\jmath+1}}{\Delta x} \cdot \frac{\vartheta_{\imath+1}^\jmath - \vartheta_\imath^\jmath}{\Delta x} + \left| \frac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^\jmath}{\Delta t} \right|^2 \right). \end{aligned} \tag{5.51}$$

Proposition 5.4.2. *Assume that $\Delta x, \Delta t \in (0, 1)$ satisfy $\Delta t \le \Delta x$. Then, the discrete system* (5.50) *dissipates energy according to:*

$$\frac{\xi_\jmath - \xi_{\jmath-1}}{\Delta t} = -\Delta x \sum_{\imath=0}^{N_x} d_\imath \left| \frac{\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1}}{2\Delta t} \right|^2, \tag{5.52}$$

for each time step $t_\jmath$, where $\vartheta_\imath^\jmath$ denotes the numerical solution of (5.50).

Proof. By multiplying the discretized system (5.50) first equation by an approximation of ϑ_t

$$\frac{\Delta x}{2\Delta t}\left(\vartheta_\imath^{\jmath+1} - \vartheta_\imath^{\jmath-1}\right)$$

and then computing the sum when i varies from 1 to N_x, we obtain

$$\begin{aligned}
&\frac{\Delta x}{2(\Delta t)^3}\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_\imath^{J+1}-2\vartheta_\imath^J+\vartheta_\imath^{J-1}\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right)\\
&-\frac{\Delta x}{2\Delta t(\Delta x)^2}\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_{\imath+1}^J-2\vartheta_\imath^J+\vartheta_{\imath-1}^J\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right)\\
&+L^2\frac{\Delta x}{2\Delta t}\sum_{\imath=1}^{\imath=N_x}\vartheta_\imath^J(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1})\\
&+\Delta x\sum_{\imath=1}^{\imath=N_x}d_\imath\left|\frac{\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}}{2\Delta t}\right|^2=0.
\end{aligned}\tag{5.53}$$

We set:

$$S_1=\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_\imath^{J+1}-2\vartheta_\imath^J+\vartheta_\imath^{J-1}\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right),\quad S_2=\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_{\imath+1}^J-2\vartheta_\imath^J+\vartheta_{\imath-1}^J\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right).$$

- As $\vartheta_{0,J}=0$ and $(a-2b+c)(a-c)=(a-b)^2-(b-c)^2$, for all $a,b,c\in\mathbb{R}$ then

$$S_1=\sum_{\imath=0}^{\imath=N_x}\left|\vartheta_\imath^{J+1}-\vartheta_\imath^J\right|^2-\sum_{\imath=0}^{\imath=N_x}\left|\vartheta_\imath^J-\vartheta_\imath^{J-1}\right|^2.\tag{5.54}$$

- Since $\vartheta_{0,J}=0$ then:

$$S_2=\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_{\imath+1}^J-\vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right)+\sum_{\imath=1}^{\imath=N_x}\left(\vartheta_{\imath-1}^J-\vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right)$$

$$=\sum_{\imath=0}^{\imath=N_x}\left(\vartheta_{\imath+1}^J-\vartheta_\imath^J\right)\left(\vartheta_\imath^{J+1}-\vartheta_\imath^{J-1}\right)-\sum_{\imath=0}^{\imath=N_x-1}\left(\vartheta_{\imath+1}^J-\vartheta_\imath^J\right)\left(\vartheta_{\imath+1}^{J+1}-\vartheta_{\imath+1}^{J-1}\right)$$

By adding and subtracting the same amount $\left(\vartheta^J_{N_x+1}-\vartheta^J_{N_x}\right)\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)$, we get

$$
\begin{aligned}
S_2 &= \left(\vartheta^J_{N_x+1}-\vartheta^J_{N_x}\right)\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)+\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left(\vartheta^{J+1}_\imath-\vartheta^{J-1}_\imath\right)\\
&\quad-\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left(\vartheta^{J+1}_{\imath+1}-\vartheta^{J-1}_{\imath+1}\right)
\end{aligned}
$$

$$
=\left(\vartheta^J_{N_x+1}-\vartheta^J_{N_x}\right)\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)+\sum_{i=0}^{i=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left[\left(\vartheta^{J+1}_\imath-\vartheta^{J-1}_\imath-\vartheta^{J+1}_{\imath+1}+\vartheta^{J-1}_{\imath+1}\right)\right]
$$

$$
\begin{aligned}
&=\left(\vartheta^J_{N_x+1}-\vartheta^J_{N_x}\right)\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)-\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^{J+1}_{\imath+1}-\vartheta^{J+1}_\imath\right)\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\\
&\quad+\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left(\vartheta^{J-1}_{\imath+1}-\vartheta^{J-1}_\imath\right).
\end{aligned}
$$

From the boundary condition in $(5.50)_2$, we get

$$
\vartheta^J_{N_x+1}-\vartheta^J_{N_x}=-\frac{m(1)\Delta x}{(\Delta t)^2}\left(\vartheta^{J+1}_{N_x+1}-2\vartheta^J_{N_x+1}+\vartheta^{J-1}_{N_x+1}\right)-L^2\Delta x\vartheta^J_{N_x+1}.
$$

Then using the same method of calculation as for S_1 in (5.53), we get

$$
\begin{aligned}
\left(\vartheta^J_{N_x+1}-\vartheta^J_{N_x}\right)\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)&=-\frac{m(1)\Delta x}{(\Delta t)^2}\left(\left|\vartheta^{J+1}_{N_x+1}-\vartheta^J_{N_x+1}\right|^2-\left|\vartheta^J_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right|^2\right)\\
&\quad-L^2\Delta x\vartheta^J_{N_x+1}\left(\vartheta^{J+1}_{N_x+1}-\vartheta^{J-1}_{N_x+1}\right)\\
&=-m(1)\Delta x\left|\frac{\vartheta^{J+1}_{N_x+1}-\vartheta^J_{N_x+1}}{\Delta t}\right|^2-L^2\Delta x\vartheta^{J+1}_{N_x+1}\vartheta^J_{N_x+1}\\
&\quad+m(1)\Delta x\left|\frac{\vartheta^J_{N_x+1}-\vartheta^{J-1}_{N_x+1}}{\Delta t}\right|^2+L^2\Delta x\vartheta^J_{N_x+1}\vartheta^{J-1}_{N_x+1}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
S_2&=-m(1)\Delta x\left|\frac{\vartheta^{J+1}_{N_x+1}-\vartheta^J_{N_x+1}}{\Delta t}\right|^2+L^2\Delta x\vartheta^{J+1}_{N_x+1}\vartheta^J_{N_x+1}-\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left(\vartheta^{J+1}_{\imath+1}-\vartheta^{J+1}_\imath\right)\\
&\quad+m(1)\Delta x\left|\frac{\vartheta^J_{N_x+1}-\vartheta^{J-1}_{N_x+1}}{\Delta t}\right|^2+L^2\Delta x\vartheta^J_{N_x+1}\vartheta^{J-1}_{N_x+1}+\sum_{\imath=0}^{\imath=N_x}\left(\vartheta^J_{\imath+1}-\vartheta^J_\imath\right)\left(\vartheta^{J+1}_{\imath+1}-\vartheta^{J-1}_\imath\right)
\end{aligned}
$$

Then injecting S_1 and S_2 in (5.53), it follows that:

$$\begin{aligned}
&\frac{m(1)}{2\Delta t}\left|\frac{\vartheta^{J+1}_{N_x+1}-\vartheta^{J}_{N_x+1}}{\Delta t}\right|^2+\frac{L^2}{2\Delta t}\vartheta^{J+1}_{N_x+1}\vartheta^{J}_{N_x+1}+L^2\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{\imath=N_x}\vartheta^{J+1}_\imath\vartheta^{J}_\imath\\
&+\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{\imath=N_x}\left(\frac{\vartheta^{J+1}_{\imath+1}-\vartheta^{J+1}_{\imath}}{\Delta x}\frac{\vartheta^{J}_{\imath+1}-\vartheta^{J}_{\imath}}{\Delta x}+\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J}_{\imath}}{\Delta t}\right|^2\right)\\
&-\frac{m(1)}{2\Delta t}\left|\frac{\vartheta^{J}_{N_x+1}-\vartheta^{J-1}_{N_x+1}}{\Delta t}\right|^2-\frac{L^2}{2\Delta t}\vartheta^{J}_{N_x+1}\vartheta^{J-1}_{N_x+1}-L^2\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{\imath=N_x}\vartheta^{J}_\imath\vartheta^{J-1}_\imath\\
&-\frac{\Delta x}{2\Delta t}\sum_{\imath=0}^{\imath=N_x}\left(\frac{\vartheta^{J}_{\imath+1}-\vartheta^{J}_{\imath}}{\Delta x}\frac{\vartheta^{J-1}_{\imath+1}-\vartheta^{J-1}_{\imath}}{\Delta x}+\left|\frac{\vartheta^{J}_{\imath}-\vartheta^{J-1}_{\imath}}{\Delta t}\right|^2\right)\\
&=-\Delta x\sum_{\imath=1}^{\imath=N_x}d_\imath\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J-1}_{\imath}}{2\Delta t}\right|^2
\end{aligned}\tag{5.55}$$

Hence

$$\frac{\xi_J-\xi_{J-1}}{\Delta t}=-\Delta x\sum_{\imath=1}^{\imath=N_x}d_\imath\left|\frac{\vartheta^{J+1}_{\imath}-\vartheta^{J-1}_{\imath}}{2\Delta t}\right|^2$$

The proof is achieved. □

5.4.1 Test Problems

In this part, a few tests in one dimension will be conducted in order to illustrate the outcome of the previously established exponential stability of system (5.1). To carry out the simulations, the following parameters are fixed:

- Domains:

$$\Omega_x=(\frac{1}{2};1),\quad \omega_x=(\frac{3}{4};1),\tag{5.56}$$

- Discretization parameters:

$$N_x=49,\quad N_t=9999,\quad T=100\,\text{seconds}.$$

- Geometric parameter: $L=1$.

The corresponding mesh sizes of the simulation grid are:

$$\Delta x=\frac{1-r_0}{N_x+1}=0.01,\quad \Delta t=\frac{T}{N_t+1}=0.01,$$

and the time t_j at which the state ϑ^j is computed is:

$$t_j = j\Delta t = \frac{j}{100}. \tag{5.57}$$

The following example illustrates the exponential stability of the system and clarifies the role of each parameter involved.

Example 5.4.3. ***(Two-Dimensional Physical Phenomenon Modeled by System** (5.48)**)***
Consider an annular membrane M, made of a synthetic material (such as a stretched elastic polymer), with inner radius $r_0 = 0.5\,m$ and outer radius $r_M = 1\,m$, stretched horizontally in a plane. The membrane is fixed at its inner boundary to a rigid disk D, centered at the origin and of radius r_0. M is also connected at its outer edge to a flexible metallic ring made of a lightweight alloy (e.g., flexible aluminum), with surface mass density σ in kg·m^{-2}, capable of undergoing transverse oscillations. The portion of the metallic ring in direct contact with the membrane has a linear mass density $\mu = m(1)$ in $kg \cdot m^{-1}$.

Between the radii $r = 0.75\,m$ and r_M, a viscoelastic material with viscosity coefficient d_0 in $kg \cdot m^2 \cdot s^{-1}$ is embedded in the membrane.

The objective is to analyze the transverse deformations $\vartheta = \vartheta((x, y), t)$ of M over a time interval $T = 1\,min\ 40$ seconds, as functions of σ and d_0, given an initial displacement ϑ_0 and an initial velocity ϑ_1.

The following two cases are considered:

$$\vartheta_0(x) = \begin{cases} \sin(\pi x), & \text{if } x \in]\frac{1}{2}, 1] \\ 0, & \text{if } x = \frac{1}{2} \end{cases}, \quad \vartheta_1(x) = 0 \quad \text{in } \overline{\Omega}_x. \tag{5.58}$$

and

$$\vartheta_0(x) = \begin{cases} x, & \text{if } x \in]\frac{1}{2}, 1] \\ 0, & \text{if } x = \frac{1}{2} \end{cases}, \quad \vartheta_1(x) = \begin{cases} e^{\frac{1}{x}} & \text{if } x \in]\frac{1}{2}, 1] \\ 0, & \text{if } x = \frac{1}{2}. \end{cases} \tag{5.59}$$

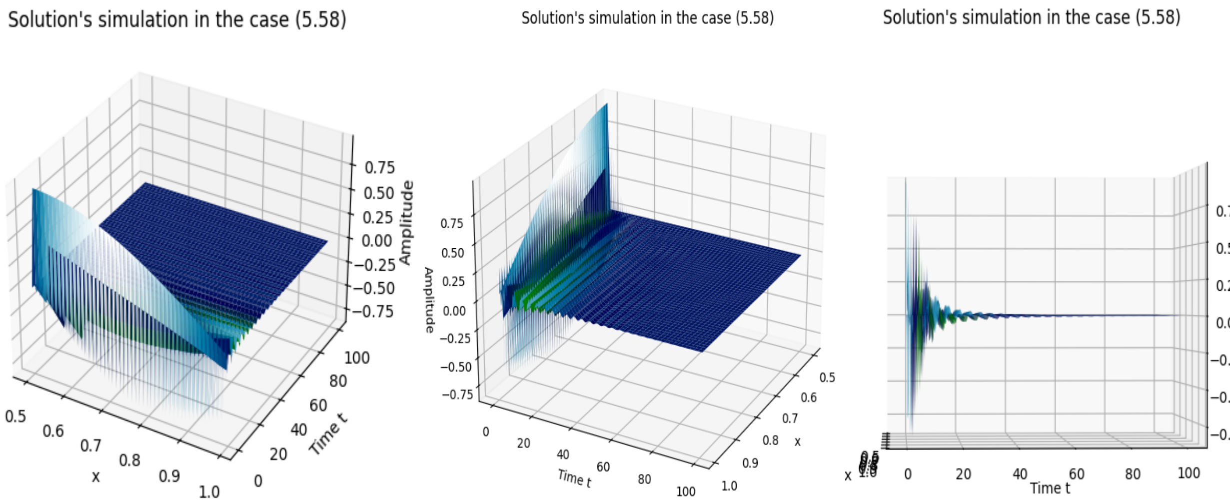

Figure 5.1: 1^{st} solution for $d_0 = m = 1$ (Global and Time views)

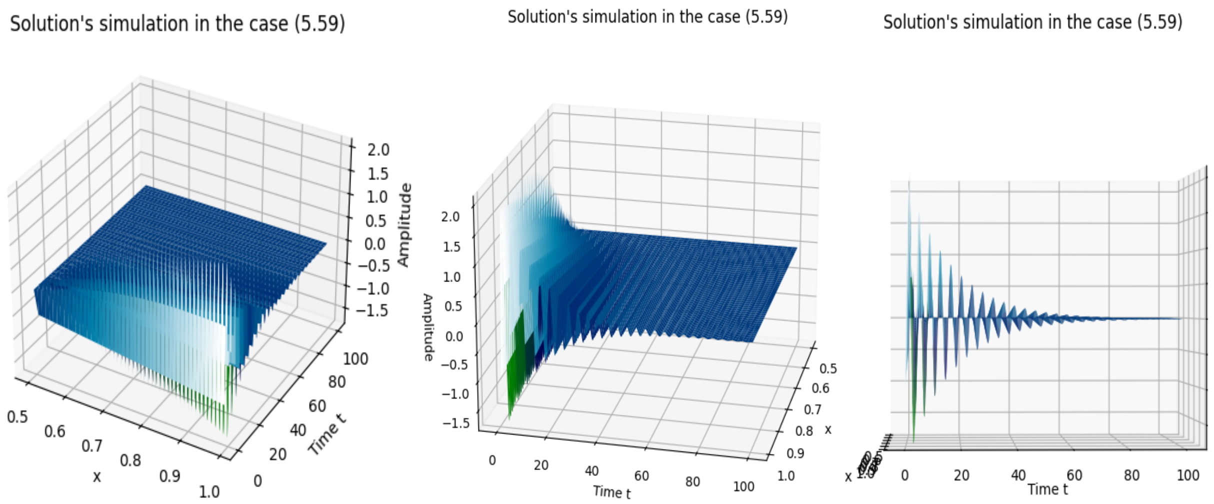

Figure 5.2: 2^{nd} solution for $d_0 = m = 1$ (Global and Time views)

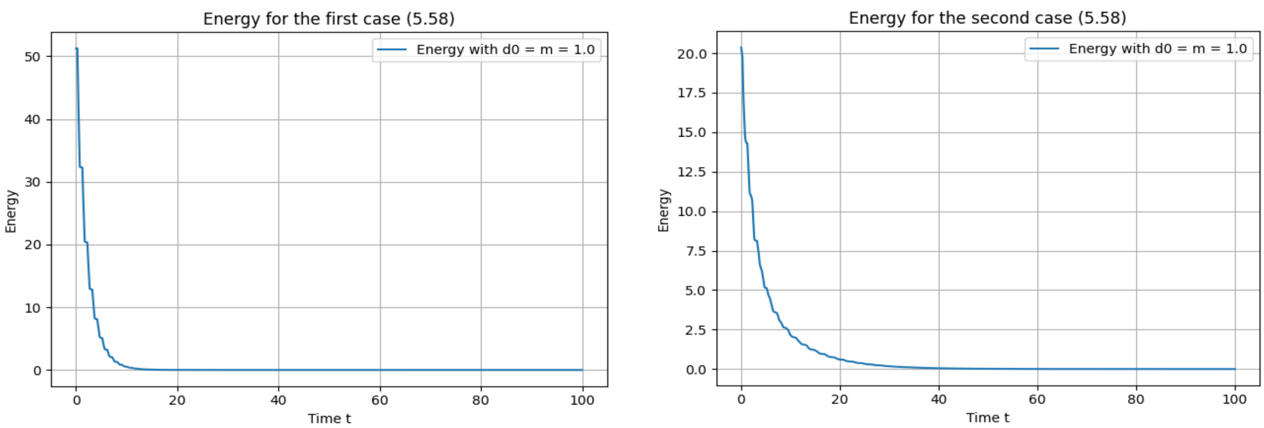

Figure 5.3: Energies of the 1^{st} and 2^{nd} solution for $d_0 = m = 1$

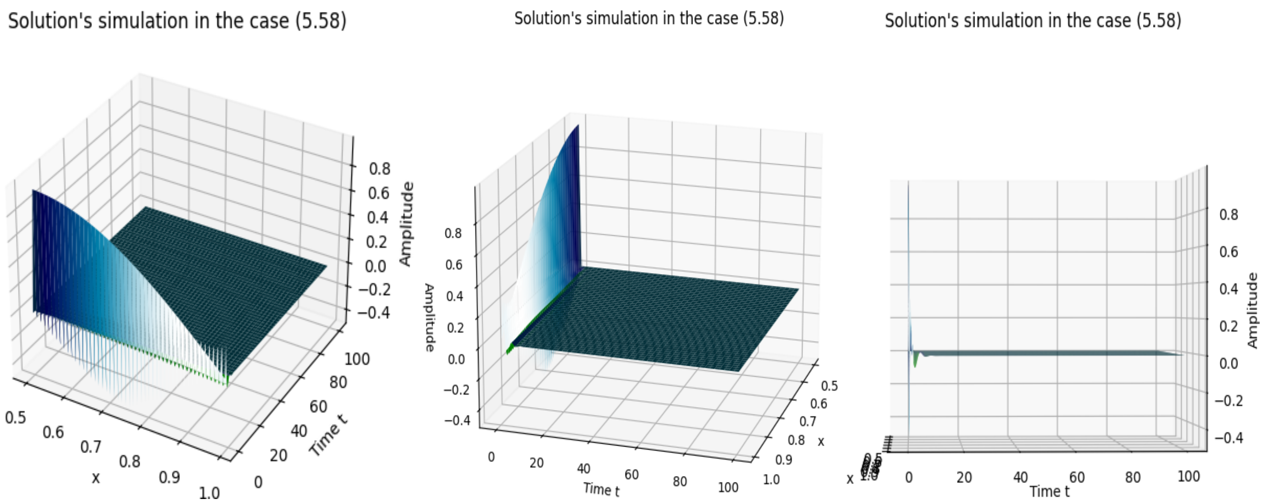

Figure 5.4: 1^{st} solution for $d_0 = 10$ and $m = 1$ (Global and Time views)

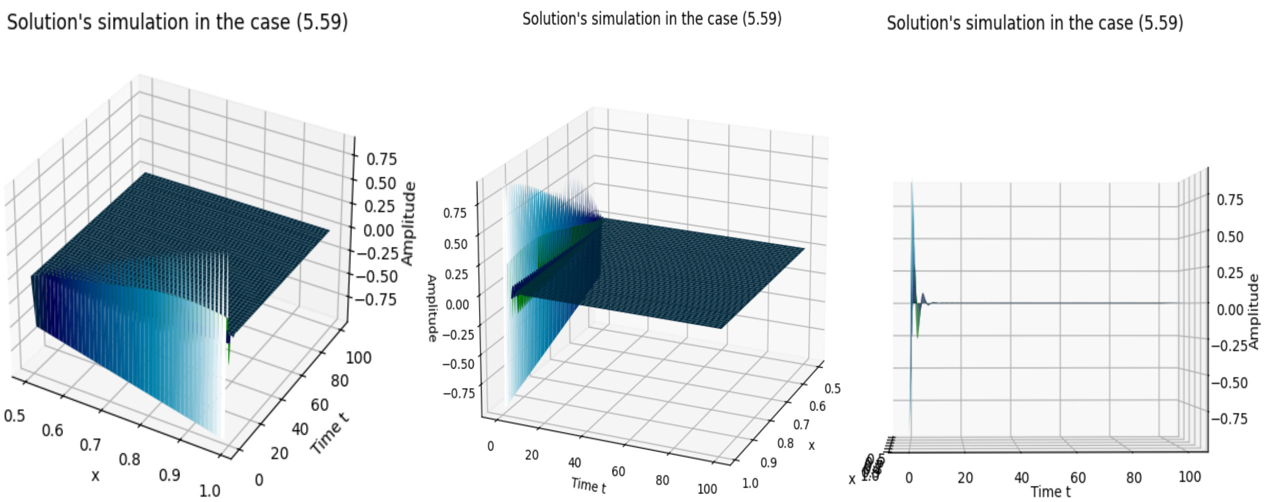

Figure 5.5: 2^{nd} solution for $d_0 = 10$ and $m = 1$ (Global and Time views)

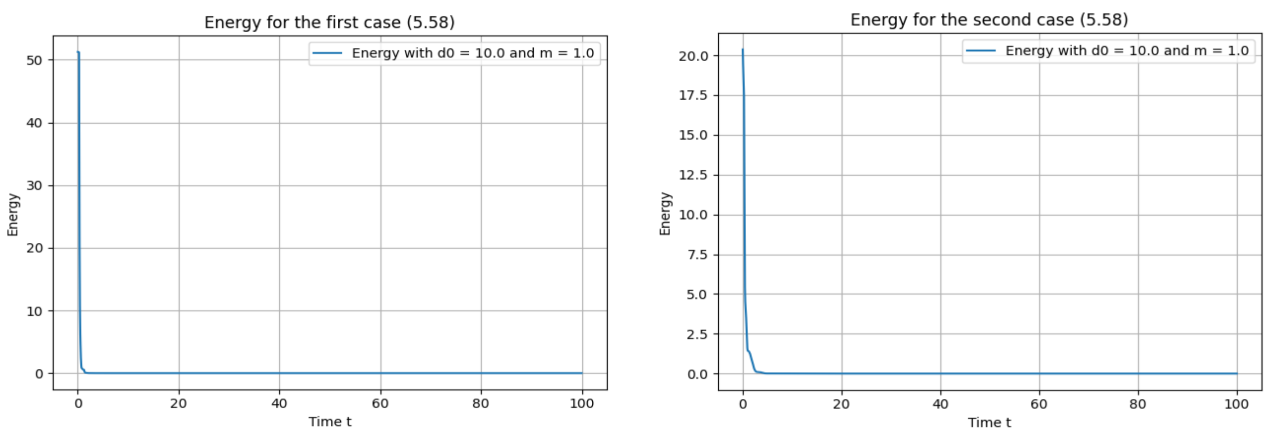

Figure 5.6: Energies of the 1^{st} and 2^{nd} solution for $d_0 = 10$ and $m = 1$

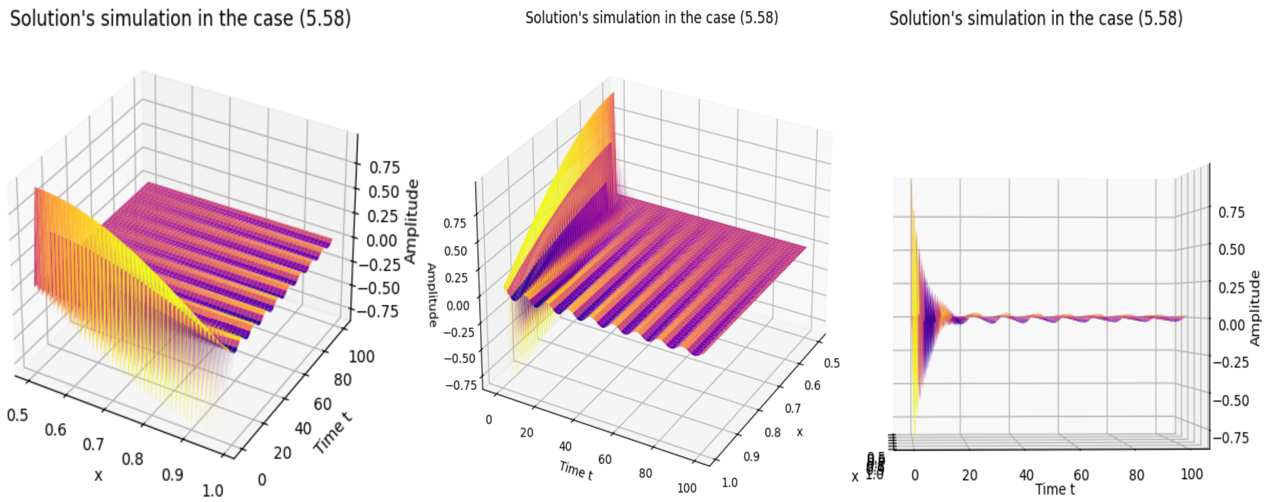

Figure 5.7: 1^{st} solution for $d_0 = 1$ and $m = 10$ (Global and Time views)

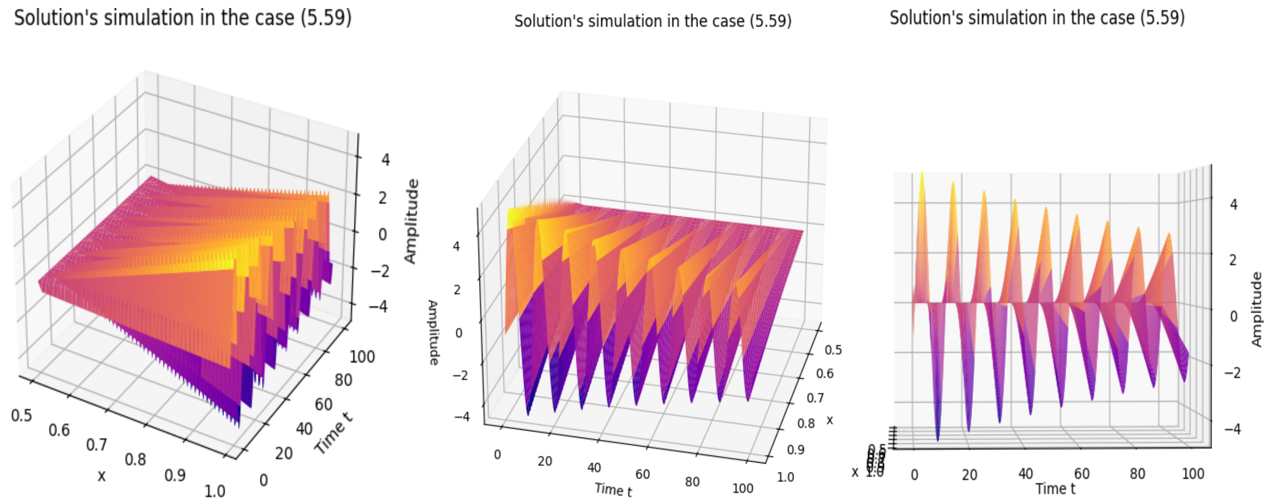

Figure 5.8: 2^{nd} solution for $d_0 = 1$ and $m = 10$ (Global and Time views)

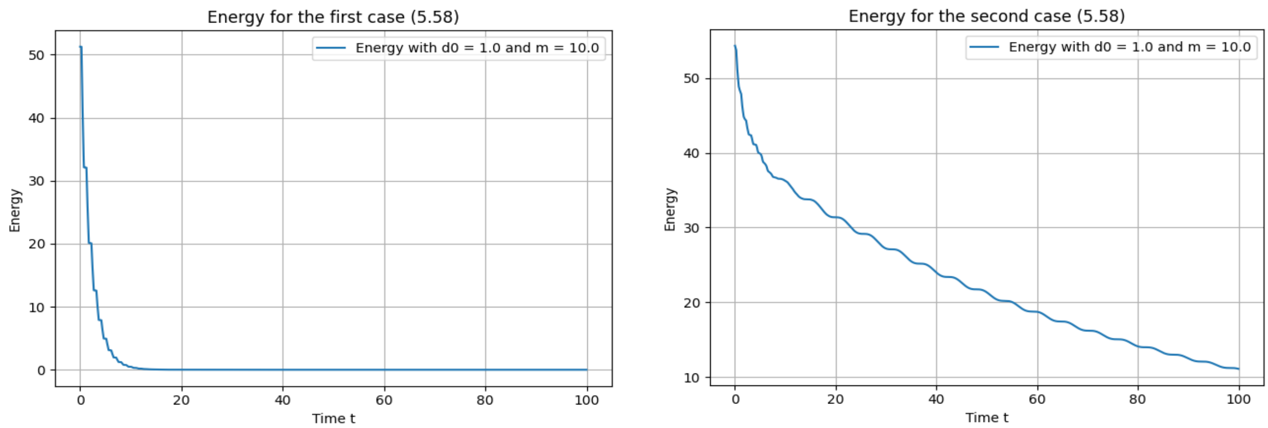

Figure 5.9: Energies of the 1^{st} and 2^{nd} solution for $d_0 = 1$ and $m = 10$

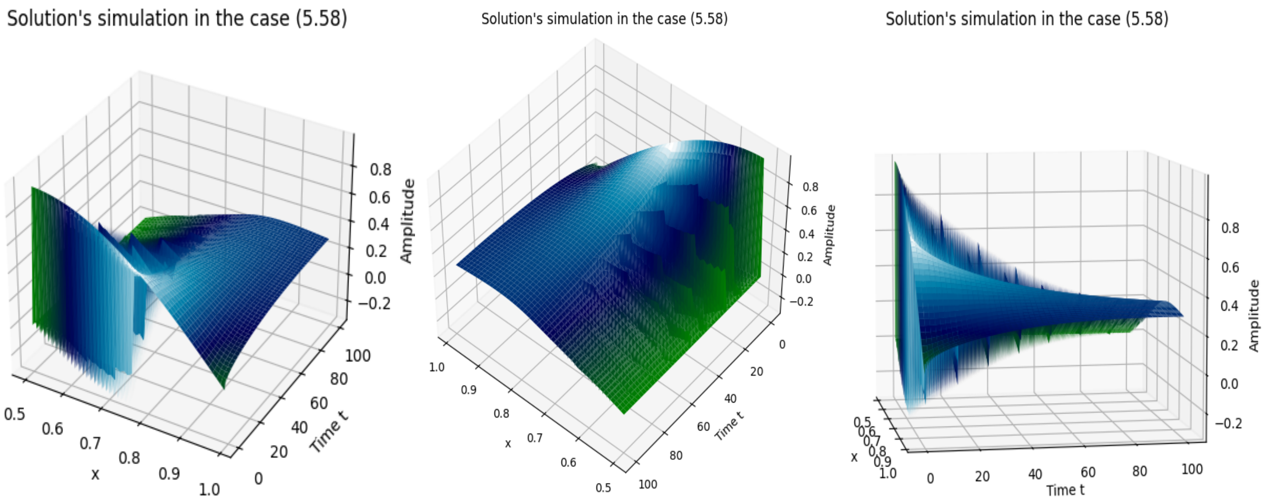

Figure 5.10: 1^{st} solution for $d_0 = 9000$ and $m = 1$ (Global and Time views)

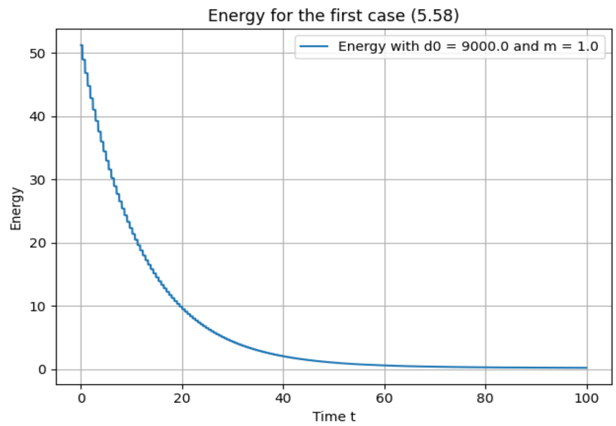

Figure 5.11: Energy of the 1^{st} solution for $d_0 = 9000$ and $m = 1$

5.4.2 Global and Temporal Analysis

To begin with, Figures 5.1, 5.2, 5.4, 5.5, 5.7, and 5.8 illustrate the consistency of the numerical scheme (5.50) with physical models of the same type, as exemplified in Example 5.4.3. These figures clearly highlight the flexibility and immediate response of the boundary at $x = 1$, which reacts promptly when $m = 1$ (fast response) and more slowly when $m = 10$ (delayed response).

In contrast, they also show a complete absence of deformation at $x = 0.5$, despite the presence of internal deformations, thereby emphasizing the rigidity of this boundary.

A rapid decay in amplitude over time is observed in cases where $d_0 > m$ or $d_0 = m = 1$. This attenuation becomes more pronounced as d_0 increases, and conversely, it diminishes when m increases. This behavior can be leveraged in practice to avoid persistent deformations, such as those illustrated in Figures 5.7 and 5.8.

Referring again to Example 5.4.3, this leads to two practical design strategies:

- Select a sufficiently thin (lightweight) metallic ring to ensure a rapid mechanical response to deformations in M;

- Alternatively, a viscoelastic material (such as dense rubber) with sufficiently high viscosity d_0 may be incorporated to reduce the sensitivity between the outer boundary and the interior of M, without fully eliminating it.

  Indeed, a complete loss of sensitivity could confine the waves within the undamped region, potentially leading to the accumulation of energy and the emergence of additional deformations (see Figure 5.10).

From a spatial perspective, the initial wave profile-sinusoidal for (5.58) and linear for (5.59) is partially preserved over time while contracting. This feature may prove useful in applications that require deformations to retain a specific structural pattern.

5.4.3 Analysis of Energies

Figures 5.3, 5.6, and 5.9 confirm the dissipative nature of the energy ξ_j, as previously established in **Proposition 5.4.2**.

Figure 5.3, corresponding to initial condition (5.58) (respectively (5.59)), displays a rapid decay of the energy toward zero, although not as pronounced as in Figure 5.6 under the same respective conditions. Since the only difference between the two cases lies in the value of the parameter d_0, this clearly highlights its influence on the energy dissipation rate. In summary, regardless of the initial condition, the greater the value of d_0, the faster the energy decays to zero.

See the tables below for a quantitative illustration of these observations:

Figure 5.3: $d_0 = m = 1$				
Initial conditions	Initial energy ξ_0	Time t_N where $\xi_N < 10^{-6}\xi_0$	Value of ξ at t_N (ξ_N)	Final energy ξ_{N_t+1}
Condition (5.58)	51.26	52.22 seconds	5.12×10^{-5}	1.47×10^{-7}
Condition (5.59)	20.36	none	none	3.73×10^{-5}

Figure 5.6: $d_0 = 10$ and $m = 1$				
Initial Conditions	Initial energy ξ_0	Time t_N where $\xi_N < 10^{-6}\xi_0$	Value of ξ at t_N (ξ_N)	Final energy ξ_{N_t+1}
Condition (5.58)	51.26	6.06 seconds	5.06×10^{-5}	7.2×10^{-59}
Condition (5.59)	20.36	9.79 seconds	2.02×10^{-5}	1.39×10^{-57}

These data highlight the effectiveness of the dissipation coefficient d_0 and of the mixed static-dynamic boundary conditions in rapidly regularizing the energy when $d_0 \geq m$, thereby ensuring rapid stabilization potentially even exponential stability.

In contrast, when $d_0 < m$ or exceeds a certain threshold, exponential stability is less likely, although the energy still decays over time, as illustrated in Figure 5.9 and 5.11.

Figure 5.9: $d_0 = 1$ and $m = 10$				
Initial conditions	Initial energy ξ_0	Time t_N where $\xi_N < 10^{-6}\xi_0$	Value of ξ at t_N (ξ_N)	Final energy ξ_{N_t+1}
Condition (5.58)	51.26	none	none	0.0008
Condition (5.59)	54.29	none	none	11.13

Figure 5.11: $d_0 = 9000$ and $m = 1$				
Initial conditions	Initial energy ξ_0	Time t_N where $\xi_N < 10^{-6}\xi_0$	Value of ξ at t_N (ξ_N)	Final energy ξ_{N_t+1}
Condition (5.58)	51.26	none	none	0.22

5.5 CONCLUSION

Two studies have been conducted to achieve exponential stability in the energy provided by the system. The Nakao method was used for theoretical analysis, revealing a need for a function characteristic d ($d < \frac{2}{c'_p}$). Numerical analysis confirmed this and another condition $d > m$. Under the combined conditions $m \leq d_0 < \frac{2}{c'_p}$, exponential stability is guaranteed. Otherwise, only asymptotic (non-exponential) stability can be ensured.

From a physical perspective, this study demonstrates that a moderate rate of energy dissipation in the damped region is necessary when dealing with locally damped waves under mixed static-dynamic boundary conditions in order to drive the system back to its rest state in a short period of time, despite the initial deformations.

In contrast, when the outer boundary is too rigid to respond effectively to incoming deformations (i.e., $d_0 < m$), or when the damping region is excessively rigid such that it prevents the outer boundary from following internal motions (d_0 excessively large), the return to the equilibrium state becomes significantly slower.

6

General Conclusion & Future Perspectives

6.1 General Conclusion

In summary, we have studied the asymptotic behavior of internal damped wave with significant/or neglected boundary kinetic energy. Although the well-posedness of those models was not a novelty in this thesis, it has been established by semi-groups and energy methods. The main difficulties was on the applicability of Nakao's method with non-homogenous Dirichlet condition, and the lack of exponential stability result under the (PMGC) condition. In chapter (3), after stating the unconditional strong stability, we have shown the exponential stability when the damped region ω is under PMGC condition. The controllability of this model thus followed from the observability result.

The chapter (4) served as prototype to the chapter (5), in term of boundary flexibility. Thus in this chapter, only the monotone decreasing role of the damper coefficient d_0 have been illustrated.

However, in chapter (5), after stating an exponential decay rate when $d_0 < \frac{2}{c_p}$, a more deeper illustration have been done. As a result, it was found that the density m has to be smaller than d_0 in order to achieve exponential stability, with the fastest possible decay rate. These results may widely help the experimental physicists in the choice of m and d_0 for such a model or even complex system where the Dirichlet condition is obsolete over the whole boundary Γ.

6.2 Future Perspectives

This research shows how the acting parameters of physical phenomenon are important. The more they are considered, the more the studied phenomenon is general and concise. In this spirit, many phenomena will be subject of my future researches, notably:

- The case where the damped function d also dissipates over time, that is, the damper gets old. The challenge here will be to see whether there exists a minimal condition on d, in order to achieve the exponential stability by Nakao's technique.

- The case where the wave is forced in Ω. Since, most of material conceptions lead to set them together and thus preventing any free vibration.

- And obviously, nonlinear cases, memory term cases, and so on.

In a more technical framework, the ambition to develop a technique inspired by that of Nakao for weaker stabilization will be the subject of future research.

Bibliography

Bibliography

[1] Adams, R. A.: Sobolev spaces, Academic press, *Pure and Applied Mathematics*, (1978)

[2] Addoun, R.I. and Laoubi, K.: Stability and regularity analysis of nonlinear wave equations with localised internal and Ventcell boundary conditions, *International Journal of Control*, **98**(4): 824-833 (2025), https://doi.org/10.1080/00207179.2024.2368052

[3] Akil, M. and Hajjej, Z.: Exponential stability and exact controllability of a system of coupled wave equations by second-order terms (via Laplacian) with only one non-smooth local damping, *Mathematical Methods in the Applied Sciences*, **47**(4): 1883-1902 (2024), https://doi.org/10.1002/mma.9724

[4] Al-Mahdi, M. and Al-Gharabi, M. M.: New general decay results in an infinite memory viscoelastic problem with nonlinear damping, *Boundary Value Probl.*, **1**(140): (2019)

[5] Al-Mahdi, M., Al-Gharabi, M. M., Kafini, M. and Al-Omri, S.: On the global existence and asymptotic behavior of the solution of a nonlinear wave equation with past history, *J. Math. Phys.*, **62**(031512): 152-156 (2021), https://doi.org/10.1063/5.0003515

[6] Al-Mahdi, A. M., et al.: Stabilization of a viscoelastic wave equation with boundary damping and a variable exponents: theoretical and numerical study, *AIMS Math*, **7**(8): 15370-15401 (2022)

[7] Alabau-Boussouira, F., Cannersa, P. and Komornik, V.: Indirect internal stabilization of weakly coupled evolution equations, *Journal of Evolution Equations*, **2**: 127-150 (2002), https://doi.org/10.1007/s00028-002-8083-0

[8] Alembert, D.: Traité de Dynamique, https://gallica.bnf.fr/ark:/12148/bpt6k15110889.image

[9] Alembert, D.: Vingt-sixième mémoire. Recherches de Calcul Intégral, p. 225-253, http://sites.mathdoc.fr/cgi-bin/oetoc?id=OE_DALEMBERT__4

[10] Amerio, L. and Prouse, G.: Abstract almost periodic functions and functional equations, *Van nostrand, Princeton, N. J.*, (1971)

[11] Amerio, L. and Prouse, G.: Zur Theorie der fastperiodische Funktionen, *Acta Math.*, **46-47**: 237-281 (1925-1926)

[12] Avila, J. A. J., Lobato, R. F. C., Chuquipoma, J. A. D. and Dos Santos, M. J.: Exponential Stability and Numerical Results of a Coupled System of Wave Equations with Indirect Control, *Appl. Math. Inf. Sci.*, **14**(3): 405-414 (2020)

[13] Barry, K. L. and Laoubi, K. : Stabilization of the wave equation with internal damping and numerical simulations, *International Journal of Control*, **1**(13): (2025), https://doi.org/10.1080/00207179.2025.2461595.

[14] Barry, K. L. and Laoubi, K.: Theoretical and numerical stabilization of the wave equation subject to internal and dynamic controllers, *Arabian Journal of Mathematics*, (2025), https://doi.org/10.1007/s40065-025-00536-w

[15] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Springer New York Dordrecht Heidelberg London*, p. 1-599 (2011), https://doi.org/10.1007/978-0-387-70914-7

[16] Burq, N. and Hitrik, M.: Energy decay for damped wave equations on partially rectangular domains, *Math. Res. Lett.*, **14**(1): 35-47 (2007), http://dx.doi.org/10.4310/MRL.2007.v14.n1.a3

[17] Cavalcanti, M. M., Cavalcanti, V. N. D., Fukuoka, R. and Toundykov, D.: Stabilization of the damped wave equation with Cauchy-Ventcel boundary conditions, *Journal of Evolution Equations*, **9**(1): 143-169 (2009), https://doi.org/10.1007/s00028-009-0002-1

[18] Cavalcanti, M., Guesmia, A. et al.: General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, *Differ. Integr. Equations*, **18**(5): 583-600 (2005)

[19] Cavalcanti, M., Khemmoudj, A. and Seghour, L.: Uniform stabilization of the damped Cauchy-Ventcel problem with variable coeficients and dynamic boundary conditions, *J. Math. Anal. Appl.*, **328**: 900-930 (2007)

[20] Cavalcanti, M. M., Lasiecka, I. and Toundykov, D.: Geometrically constrained stabilization of wave equations with Wentzell boundary conditions, *Applicable Analysis*, **91**(8): 1427-1452 (2012), https://doi.org/10.1080/00036811.2011.647910

[21] Chen, G.: A note on the boundary stabilization of wave equation, *SIAM. Journal on Control and Optimization*, **19**: 106-113 (1981)

[22] Chen, G.: Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain, *J. Math. Pures Appl.*, **58**: 249-273 (1979)

[23] Ciarlet, P. G.: Introduction à l'analyse numérique matricielle et à l'optimisation, *Masson*, (1982)

[24] Datko, R., Lagnèse, J. and Polis, M. P.: An Example on the Effect of Time Delays in Boundary Feedback Stabilization of Wave Equations, *SIAM Journal on Control and Optimization*, **24**: 152-156 (1986)

[25] Donkin, W. F.: Acoustics, *Bartholomew Price*, **1**: (1870), https://archive.org/details/cu31924031295623/page/n120/mod2/1up

[26] Douglas, C. G.: Physique Générale: Ondes, optique et physique moderne, p. 488, (1993), https://books.google.fr/books?id=tAJIt_NA3dYC&pg=PA20#v=onepage&q&f=false

[27] Engel, K. J. and Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, *Alfred A. Knopf*, (1995)

[28] Engel, K. J., Nagel, R. and Brendle, S.: One-parameter semigroups for linear evolution equations, *Springer*, **194**: (2000)

[29] Engelsman Steven, B.: D'Alembert et les équations aux dérivées partielles, *In: Dix-huitième Siècle*, (16): 27-37 (1984), https://doi.org/10.3406/dhs.1984.1480

[30] Erdös, L.: Banah-Alaglu theorem, (2007), https://www.karlin.mff.cuni.cz/~prazak/uceni/Odr2/Prednaska/alaoglu.pdf

[31] Fu, X. and Kong, L.: Longtime Behavior of Wave Equation With Kinetic Boundary Condition, *Applied Mathematics & Optimization*, **84**(3): 2803-2817 (2021), https://doi.org/10.1007/s00245-020-09730-y

[32] Fu, X. and Lü, Q.: Stabilization of the Weakly Coupled Wave-Plate System with One Internal Damping, *Vietnam Journal of Mathematics*, **49**(3): 767-786 (2021), https://doi.org/10.1007/s10013-021-00493-9

[33] Gerbi, S., Kassem, C., Mortada, A. and Wehbe, A.: Numerical Study of the Stabilization of 1D Locally Coupled Wave Equations, *Z. Anal. Anwend.*, **40**(2): 131-151 (2021), https://doi.org/10.4171/ZAA/1676

[34] Goldstein, G. R.: Derivation and physical interpretation of general boundary conditions, *Advances in Differential Equations*, **11**(4): 457-480 (2006), https://doi.org/10.57262/ade/1355867704

[35] Guesmia, A.: Asymptotic stability of abstract dissipative systems with infinite memory, *J. Math. Anal. Appl.*, **382**(2): 748-760 (2011)

[36] Guesmia, A.: New general decay rates of solutions for two viscoelastic wave equations with infinite memory, *Math. Modell. Anal.*, **25**(3): 351-373 (2020)

[37] Guesmia, A. and Messaoudi, S. A.: A general decay result for a viscoelastic equation in the presence of past and finite history memories, *Non linear Anal. : Real World Appl.*, **13**(1): 476-485 (2012)

[38] Guesmia, A. and Messaoudi, S. A.: A new approach to the stability of an abstract system in the presence of infinite history, *J. Math. Anal. Appl.*, **416**(1): 212-228 (2014)

[39] Guesmia, A. and Tatar, N. E.: Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay, *Commun. Pure Appl. Anal.*, **14**(2): 457-491 (2015)

[40] Hadamard, J. M.: Sur les problèmes aux dérivées partielles et leur signification physique, *Princeton University Bulletin*fbtxe,, **13**(4): 49-52 (1902)

[41] Haraux, A.: Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. Portugaliae mathematica, **46**(3): 245-258 (1989)

[42] Heminna, A.: Stabilisation frontière de problèmes de Ventcel, *ESAIM: COCV*, **5**: 591-622 (2000), https://doi.org/10.1051/cocv:2000123

[43] Heminna, A., Nicaise, S. and Sene, A.: Stabilisation d'un système de la thermoélasticité anisotrope avec feedbacks non linéaires, *C. R. Acad. Sci. Paris, Ser. I*, **339**: 561-566 (2004), https://doi.org/10.1016/j.crma.2004.08.008

[44] Huang, F. L.: Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Diff. Equations*, **1**(1): 43-56 (1985), https://cir.nii.ac.jp/crid/1570572700285814784

[45] Ihaddadene, L.: Décroissance générale de l'énergie pour un problème de Wentzell avec dissipation et retards non linéaires, *Thèse de doctorat USTHB, soutenue le 27/09/2021*

[46] Khemmoudj, A. and Medjen, M.: Exponential Decay for the Semilinear Cauchy-Ventcel Problem with Localized Damping, *Bol. Soc. Paran. Mat.*, **22**(2): 97-116 (2004), https://doi.org/10.5269/bspm.v22i2.7486

[47] Khemmoudj, A. and Medjen, M.: Exponential Decay for the Semilinear Cauchy-Ventcel Problem with Localized Damping, *Bol. Soc. Paran. Mat. (3s.)*, **22**(2): 97-116 (2004)

[48] Khemmoudj, A. and Seghour, L.: Exponential stabilization of a viscoelastic wave equation with dynamic boundary conditions, *Nonlinear Differ. Equ. Appl.*, **22**: 1259-1286 (2015)

[49] Komornik, V.: Exact controllability and stabilization, *RAM Research in Applied Mathematics, Masson*, p. 156 (1994)

[50] Komornik, V.: Exact Controllability and Stabilization: The Multiplier Method, *Wiley-Masson Series Research in Applied Mathematics , Wiley, Wiley*, (1995), https://books.google.dz/books?id=AmVqQgAACAAJ

[51] Komornik, V.: Exact controllability in short time for the wave equation, *Ann. de l'I.H.P., Anal. non linéaire*, **6**: 153-164 (1989)

[52] Lagnèse, J.: Decay of solutions of wave equations in a bounded region with boundary dissipation, *J. Diff. Eq.*, **50**: 163-182 (1983)

[53] Laoubi, K. and Seba, D.: Analysis of energy dissipation in hyperbolic problems influenced by internal and boundary control mechanisms, *International Journal of Control*, **98**(6): 1263-1271 (2025), https://doi.org/10.1080/00207179.2024.2390885

[54] Laoubi, K. and Seba, D.: Polynomial Energy Decay Rate for Dissipative Wave Equations with Mixed Boundary Conditions, *Acta Applicandae Mathematicae*, **169**: 629-646 (2020)

[55] Laoubi, K. and Seba, D.: Polynomial Energy Decay Rate for the Wave Equation with Kinetic Boundary Condition, *Acta Applicandae Mathematicae*, **191**(1): 1 (2024), https://doi.org/10.1007/s10440-024-00650-5

[56] Lasiecka, I. and Tataru, D.: Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping, *Differential and Integral Equations*, **6**(3): 507-533 (1993), https://doi.org/10.57262/die/1370378427

[57] Lasiecka, I. and Triggiani, R.: Uniform exponential decay in a bounded region with $L^2(0,T,L^2(\Sigma))$ feedback control in the Dirichlet boundary conditions, *J. Diff. Eq.*, **66**: 340-390 (1987)

[58] Lax, P. D., Morawetz, C. S., Phillips, R. S.: The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle, *Appl. Math*, (16): 477-489 (1963)

[59] Le Rousseau, J. and Lebeau, G.: On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations, *ESAIM: COCV*, **18**(3): 712-747 (2012), https://doi.org/10.1051/cocv/2011168

[60] Lemrabet, K.: Étude de divers problèmes aux limites de Ventcel d'origine physique ou mécanique dans des domaines non réguliers, *USTHB, Alger*, (1987)

[61] Lemrabet, K.: Problème de Ventcel pour le système de l'elasticité dans un domaine de $\mathbb{R}^3$, *Compte rendu de l'Académie des sciences, Série 1, Mathématics, Paris*, **304**: 151-154 (1987)

[62] Lemrabet, K. and Teniou, D. E.: Un problème d'évolution de type Ventcel (an evolution problem of Ventcel type), *Rev. Maghr. Math.*, **1**: 15-29 (1992)

[63] Li, C., Liang, J. and Xiao, T. J.: Dynamical behaviors of solutions to nonlinear wave equations with vanishing local damping and Wentzell boundary conditions, *Zeitschrift für angewandte Mathematik und Physik*, **69**(4): 102 (2018), https://doi.org/10.1007/s00033-018-0996-8

[64] Lions, J. L.: Contrôlabilité exacte des systèmes distribués, *C.R. Acad. Sci. Paris. série I Math.*, **302**: 471-475 (1986)

[65] Lions, J. L.: Exact Controllability, Stabilization and Perturbations for Distributed Systems. Tome 1. Contrôlabilité exacte, *Rech. Math. Appl. 8, Masson, Paris*, (1988)

[66] Lions, J. L.: The John von Neumann Lecture, *SIAM National Meeting, Boston, U.S.A.*, (1986)

[67] Liu, K.: Locally distributed control and damping for the conservative systems, *SIAM Journal on Control and Optimization*, **35**(5): 1574-1590 (1997), https://doi.org/10.1137/S0363012995284928

[68] Liu, Z. and Rao, B.: Characterization of polynomial decay rate for the solution of linear evolution equation, *Zeitschrift für angewandte Mathematik und Physik ZAMP*, **56**(4): 630-644 (2005), https://doi.org/10.1007/s00033-004-3073-4

[69] Liu, Z. and Zheng, S.: Semigroups Associated with Dissipative Systems, *Chapman & Hall*, (1999)

[70] Matsumura, A.: Energy Decay of Solutions of Dissipative Wave Equations, *Proc. Japan Acad. Serie A*, **53**: 232-236 (1977)

[71] Mizohata, S. and Mochizuki, K.: On the principle of limiting amplitude for dissipative wave equations, *J. Math. Kyoto Univ.*, **6**(1): 109-127 (1966)

[72] Mochizuki, K.: Scattering Theory for Wave Equations with Dissipative Terms, *Publ. RIMS, Kyoto Univ*, **12**: 383-390 (1976)

[73] Morawetz, C. S., Ralston, J. V. and Strauss, W. A.: Decay of solutions of the wave equation outside nontrapping obstacles, *Comm. Pure Appl. Math.*, **30**: 447-508 (1977)

[74] Münch, A.: Optimal Internal Dissipation of a Damped Wave Equation Using a Topological Approach, *Int. J. Appl. Math. Comput. Sci.*, **19**(1): 15-38 (2025), https://doi.org/10.2478/v10006-009-0002-x

[75] Nakao, M.: Asymptotic Stability of the bounded or Almost Periodic Solution of the Wave Equation with the Nonlinear Dissipative Term, *J. Math. Ana. Appl.*, **58**: 336-343 (1977)

[76] Nakao, M.: Stabilization of Local Energy in an Exterior Domain for the Wave Equation with a Localized Dissipation, *journal of differential equations*, **148**(DE983468): 388-406 (1998)

[77] Nicaise, S.: Stability and controllability of and abstract evolution equation of hyperbolic type and concrete applications, *Rendiconti di Matematica, Serie VII*, **23**: 83-116 (2003)

[78] Pereira, D. C., Raposo, C. A. and Avila, J. A. J.: Numerical solution and exponential decay to Von Kármán system with frictional damping, *Appl. Math. Inf. Sci.*, **8**(4): 1575-1582 (2014)

[79] Phung, K. D.: Polynomial decay rate for the dissipative wave equation, *Journal of Differential Equations*, **240**(1): 92-124 (2007), https://www.sciencedirect.com/science/article/pii/S0022039607001751

[80] Poquelin, G.: Géométrie nodale et valeurs propres de l'opérateur de Laplace et du p-laplacien, *Thèse de doctorat Université de Montréal, soutenue en septembre 2015, acceptée le 6 novembre 2015*, p. 1-94 (2015), https://core.ac.uk/download/pdf/151553921.pdf

[81] Prüss, J.: On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.*, **284**(2): 847-857 (1984)

[82] Rauch, J. and Taylor, M.: Decaying States of Perturbed Wave Equations, *J. Math. Anal. Appl.*, **54**: 279-285 (1976)

[83] Russell, D. L.: A General Framework for the Study of Indirect Damping Mechanisms in Elastic Systems, *J. Math. Anal. Appl.*, **173**(2): 339-358 (1993), https://doi.org/10.1006/jmaa.1993.1071

[84] Russell, D. L.: Exact boundary value controllability theorems for wave and heat processes in star-complemented regions, *in differential game and control theory, Raxin. Liu and Sternberg, Eds, Maervel Dekker Inc, New York*, (1974)

[85] Schmidt, C.: Sur l'origine du Principe Général de Jean Le Rond D'Alembert, *Annals of Science*, **70**(4): 493-530, https://doi.org/10.1080/00033790.2012.752525

[86] Strauss, W. A.: Dispersal of waves vanishing on the boundary of an exterior domain, *Comm. Pure Appl. Math.*, **28**: 265-278 (1975)

[87] Suh, I. H. and Bien, Z.: Use of time delay action in the controller design, *IEEE Transaction of Automatic Control*, **25**: 600-603 (1980)

[88] Tcheugoué Tebou, L.: Stabilization of the damped wave equation with Cauchy-Ventcel boundary conditions, *Comptes Rendus Mathematique*, **342**(11): 859-864 (2006), https://doi.org/10.1016/j.crma.2006.04.010

[89] Uesaka, H.: The total energy decay of solutions for the wave equation with a dissipative term, *J. Math. Kyoto U niv. (JMKYAZ)*, *20*(1): 57-65 (1979)

[90] Venttsel' A. D.: On Boundary Conditions For Multidimensional Diffusion Processes, *Teor. Veroyatnost. i Primenen*, **4**(2): 172-185 (1959), https://www.mathnet.ru/eng/tvp4877

[91] Vicente, A. and Frota, C. L.: Uniform stabilization of wave equation with localized damping and acoustic boundary condition, *Journal of Mathematical Analysis and Applications*, **436**(2): 639-660 (2016), https://doi.org/10.1016/j.jmaa.2015.12.039

[92] von Kármán, T.: Festigkeitsprobleme im Maschinenbaum, *Encyklopadie der Math. Wiss. V/4C, Leipzig*, p. 311-385 (1910)

[93] Ward, C. and Kincaid, D.: Numerical mathematics and computing, *International Thomson Publishing*, (1998)

[94] Weisstein, W.: Banah-Alaglu theorem, https://mathworld.wolfram.com/Banach-AlaogluTheorem.html

[95] Wilcox, C.: The inatial-value problem for the wave equation in an exterior domain with spherical boundary, *Notices Amer. Math. Soc.*, **6 Abstract#**(564-240): 869-870 (1959)

[96] Yamada, Y.: On some quasilinear wave equations with dissipative terms, *Nagoya Math. J.*, **87**: 17-39 (1982)

[97] Zhang, X. and Zuazua, E.: Long-Time Behavior of a Coupled Heat-Wave System Arising in Fluid-Structure Interaction, *Archive for Rational Mechanics and Analysis*, **184**(1): 49-120 (2007), https://doi.org/10.1007/s00205-006-0020-x

[98] Zwillinger, D. and Dobrushkin, V.: Handbook of differential equations, *Chapman and Hall/CRC*, (2021)