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A novel method for fractal-fractional differential equations

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Abstract We consider the reproducing kernel Hilbert space method to construct numerical solutions for some basic fractional ordinary differential equations (FODEs) under fractal fractional derivative with the generalized Mittag–Leffler (M-L) kernel. Deriving the analytic and numerical solutions of this new class of differential equations are modern trends. To apply this method, we use reproducing kernel theory and two important Hilbert spaces. We provide three problems to illustrate our main results including the profiles of different representative approximate solutions. The computational results are compared with the exact solutions. The results obtained clearly show the effect of the fractal fractional derivative with the M-L kernel in the obtained outcomes. Meanwhile, the compatibility between the approximate and exact solutions confirms the applicability and superior performance of the method.

1. Introduction

Fractional calculus (FC) is one of the essential subjects of study that magnetizes researchers' interest in widespread fields of fluid mechanics, finance, wave propagation, groundwater, and visco-elasticity [24,36]. FC is regarded as the generalization of the ordinary derivative and integral concept. Differential and integral operators have gained immense importance over the years due to their ability to model many real-world problems in a way that makes the nature within humankind live understandable, controllable, and predictable [10,15,19,25,26,28,30–35,43,50–52]. In the relevant literature, there exist different types of differintegrals operators: the classical differintegrals operators, differintegrals operators based on the generalized M-L function, differintegrals operators based on power-law process, and differintegrals operators based on the exponential decay law. Recent researchers have concentrated on introducing new theories, giving more innovative ideas, and discovering various applications concerning the

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three mentioned differintegrals operators [5,7,22,41,44,46]. The spotlight in this study is on some new differintegrals operators called fractal-fractional differential and integral operators. These new operators are newly proposed by Atangana [8]. The idea behind these new operators is to combine the fractal operator with the fractional differintegrals concept where the fractional operator is considered in Caputo's sense, Caputo-Fabrizio's sense, and Atangana-Baleanu's sense. And, the fractional derivative which is based on the generalized M-L function is what we are concerned with in our work. This new operator (The ABC derivative) has all the benefits of that Caputo and Fabrizio as well as it is very useful in thermal science and material sciences, see [9,43] for more information of the benefits of the mentioned new class of operators. Unlike the other differintegrals operators, which don't fill the gap that exists to control humankind's environment, the latter new operators (fractal-fractional differential and integral operators) help researchers to understand various natural behaviors that encounter them. Also, these new operators can much better capture nature's complexities. For further details, see [9,11,23,37,38] and the references therein. Such models help researchers to understand and predict nature if their solutions are available. But how can they solve them? The answer is that there are two main ways to do this: Certain models can be solved by using the analytical methods that give their exact solutions, but many others cannot. This problem leads researchers to use numerical methods for obtaining their numerical solutions. The list of these methods is endless. The numerical schema based on the variational iteration algorithm-I [2,3], the local meshless method [20], the $(1/G\prime)$-expansion method [48], and the modified variational iteration algorithm-II [4,1] are enough to support this point. For more research papers concerning analytical and numerical studies for various fractional problems, we refer to [13,18,27,42,45,47].

The new class of fractal-fractional differential and integral operators gave birth to a new class of integral and differential equations in which there is not much work in the literature on the use of these new operators. The current paper is motivated, then, by the work of Atangana [9]. The main aim of this latter research is to gain the exact solution of some new differential equations using the properties of the Laplace transform. To our knowledge, there is no research where the reproducing kernel Hilbert space method (RKHSM) is applied to a fractal-fractional differential equation. Hence in this research and for the first time, we apply the RKHSM to solve some important FODEs under fractal fractional derivative with the M-L kernel in which these equations have exact solutions. In this process, we convert the considered problem to another one by using several steps. In consequence, by applying the RKHSM to the converted problem, the numerical solution is derived.

The Reproducing kernel theory, which was first proposed by Zaremba [49], has been extensively employed to solve fractional differential equations numerically. The representative of the reproducing kernel theory is the RKHSM. Two important advantages of this method are the uniform convergence between the numerical solutions and exact solutions and also between their derivatives. The other advantage is that this method is mesh-free. And this means that it doesn't need much work to apply it. The RKHSM has been applied by many researchers to deal with the numerical solutions of different (classical or fractional) differential and integral equations like fractional Bagley-Torvik equation [40], Riccati differential equations [39], fractional Bratu-type equations [12], nonlinear fractional Volterra integro-differential equations [21], nonlocal fractional boundary value problems [17], forced Duffing equations with nonlocal boundary condition [16].

One of our motivations is to exploit the efficient and simple features of the new fractal-fractional operators with the generalized M-L kernel to gain the approximate solutions to the proposed problems. We applied the RKHSM that is based on the RK theory. The second one is that the performance and applicability of the RKHSM are tested on the considered problems.In the next section, some basic concepts and definitions are presented to facilitate the method's application. In Section 3, the RKHSM theoretical results with the solution's representation are discussed. In Section 4, numerical experiments are provided to show the method's effectiveness. This paper is finished with the concluding paragraph.

2. The fractional derivative operators

In this section, we will provide some useful details connected to the new definitions of the fractal fractional derivatives (FF) which are introduced very recently.

Definition 2.1. Let $\mathfrak{f} \in C^1_{-1}[t_0, T], T \in \mathbb{R}^+_*$. A fractional derivative of order $\gamma \in (0,1)$ of the function $\mathfrak{f}$ that given below by [36]

$$ {}^{t_0}_{C}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \frac{1}{\Gamma(1-\gamma)} \int_{t_0}^{\eta} \mathfrak{f}'(\zeta)(\eta-\zeta)^{-\gamma} \mathrm{d}\zeta, \tag{1} $$

is called the Caputo fractional derivative.

Definition 2.2. A fractional derivative of order γ of the function $\mathfrak{f}$ (not necessary differential) that given below by [36]

$$ {}^{t_0}_{RL}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}\eta} \int_{t_0}^{\eta} \mathfrak{f}(\zeta)(\eta-\zeta)^{-\gamma} \mathrm{d}\zeta, \qquad 0 < \gamma \leq 1, \tag{2} $$

is called the Riemann–Liouville (RL) fractional derivative.

Here, $AB(\gamma)$ is a normalization function given by $AB(\gamma) = 1 - \gamma + \frac{\gamma}{\Gamma(\gamma)}$ in which $AB(0) = AB(1) = 1$. While, $E_\gamma(y) = \sum_{i=0}^{\infty} \frac{y^i}{\Gamma(\gamma i + 1)}$ is the M-L function.

Definition 2.3. Let A function $\mathfrak{f}$ be continuous and differentiable in an open interval $(t_0, T), T \in \mathbb{R}^+_*$. The FF derivative operator of $\mathfrak{f}$ of order γ in RL sense with the generalized M-L kernel is given by [9]

$$ {}^{t_0}_{FFM}D^{\eta}_{\gamma,\beta}\mathfrak{f}(\eta) = \frac{AB(\gamma)}{1-\gamma} \frac{\mathrm{d}}{\mathrm{d}\eta^{\beta}} \int_{t_0}^{\eta} \mathfrak{f}(\zeta) E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right) \mathrm{d}\zeta, \quad 0 < \gamma, \beta \leqslant 1. \tag{3} $$

In the latter formula, it is required that $\mathfrak{f}$ is fractally differentiable in (t_0, T) with order β.

Definition 2.4. Let $\gamma \in [0, 1]$ and $\mathfrak{f} \in H^1(t_0, T), T > t_0$ in which $H^1(t_0, T)$ denotes the Sobolev space. The following fractional derivative of order γ of a function $\mathfrak{f}$ [6]

$$ {}_{ABC}^{t_0}D_\gamma^\eta \mathfrak{f}(\eta) = \frac{AB(\gamma)}{1-\gamma}\int_{t_0}^{\eta} \mathfrak{f}'(\zeta)E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^\gamma\right)\mathrm{d}\zeta \tag{4} $$

is called the Atangana-Baleanu fractional derivative in the Caputo sense (ABC).

Definition 2.5. Let $\gamma \in [0,1]$ and $\mathfrak{f} \in H^1(t_0, T), T > t_0$. The following fractional derivative of order γ of a function $\mathfrak{f}$ (not necessary differential) [6]

$$ {}_{ABR}^{t_0}D_\gamma^\eta \mathfrak{f}(\eta) = \frac{AB(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_{t_0}^{\eta} \mathfrak{f}(\zeta)E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^\gamma\right)\mathrm{d}\zeta \tag{5} $$

is called the Atangana-Baleanu fractional derivative in the RL sense (ABR).

Definition 2.6. The Atangana-Baleanu fractional integral of order γ is characterized as [6]

$$ {}_{AB}^{t_0}I_\gamma^\eta \mathfrak{f}(\eta) = \frac{1-\gamma}{AB(\gamma)}\mathfrak{f}(\eta) + \frac{\gamma}{AB(\gamma)\Gamma(\gamma)}\int_{t_0}^{\eta} \mathfrak{f}(\zeta)(\eta-\zeta)^{\gamma-1}\mathrm{d}\zeta. \tag{6} $$

Definition 2.7. Let $u(t)$ be described for $t \geqslant 0$. The Laplace transform of $u(t)$ given by $U(s)$ or $L\{u(t)\}$, is an integral transform given by the Laplace integral [9]:

$$ L\{u(t)\} = U(s) = \int_0^\infty \exp(-st)u(t)dt. \tag{7} $$

Definition 2.8. We give the Laplace transform of the ABC fractional derivative by [9]

$$ \mathscr{L}\left({}_{ABC}^{0}D_\gamma^\eta \mathfrak{f}(\eta)\right) = (s\mathscr{L}(\mathfrak{f}(\eta)) - \mathfrak{f}(0))\frac{AB(\gamma)s^{\gamma-1}}{s^\gamma(1-\gamma)+\gamma}. \tag{8} $$

Remarks.

1. For the Atangana-Baleanu derivative, we have the following useful relation [9]:

$$ {}_{ABC}^{0}D_\gamma^\eta \mathfrak{f}(\eta) = \mathfrak{f}'(\eta) * \frac{AB(\gamma)}{1-\gamma}E_\gamma\left(-\frac{\gamma}{1-\gamma}\eta^\gamma\right). \tag{9} $$

2. In particular, when $0 < \gamma < 1$, there is a relationship between the ABC and ABR derivatives that is represented by [9]:

$$ {}_{ABC}^{t_0}D_\gamma^\eta \mathfrak{f}(\eta) = {}_{ABR}^{t_0}D_\gamma^\eta \mathfrak{f}(\eta) - \frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(t_0)E_\gamma\left(-\frac{\gamma}{1-\gamma}\eta^\gamma\right). \tag{10} $$

3. The reproducing kernel theory

In this part, we start with recalling some basic analysis of the reproducing kernel Hilbert space (RKHS). From here, we denote $[0,T] = \mathscr{I}, T \in \mathbb{R}_*^+$.

Definition 3.1. A function $\mathfrak{F} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ is called a reproducing kernel of the space $\mathfrak{H}$ if it satisfies [14]

1. $\mathfrak{F}(\cdot,\eta) \in \mathfrak{H}$, $\forall \eta \in \mathfrak{X}$.
2. (Reproducing property.) $\langle \mathfrak{f}, \mathfrak{F}(\cdot,\eta)\rangle = \mathfrak{f}(\eta), \forall \mathfrak{f} \in \mathfrak{H}$ and $\forall \eta \in \mathfrak{X}$.

Where $\mathfrak{H}$ is a Hilbert space over a nonempty set $\mathfrak{X}$.

Remark.

- A Hilbert space $\mathfrak{H}$ that has a reproducing kernel function $\mathfrak{F}$ (RKF) is called a "reproducing kernel Hilbert space".

Throughout "Absolutely Continuous" is abbreviated as "a. c.".

Definition 3.2. Given an interval $\mathscr{I} \subset \mathbb{R}$ and $\mathfrak{f} : \mathscr{I} \to \mathbb{R}$. A function space $\mathscr{W}_2^2(\mathscr{I})$ is [39]

$$ \mathscr{W}_2^2(\mathscr{I}) = \{\mathfrak{f}(\eta) | \mathfrak{f}^{(\mathfrak{i})} \text{ are a.c. functions on } \mathscr{I}, \mathfrak{f}'' \in L^2(\mathscr{I}), \mathfrak{i} = 0, 1, \text{ and } \mathfrak{f}(0) = 0\}. $$

with the inner product:

$$ \langle \mathfrak{f}, \mathfrak{g}\rangle_{\mathscr{W}_2^2} = \sum_{\mathfrak{i}=0}^{1}\mathfrak{f}^{(\mathfrak{i})}(0)\mathfrak{g}^{(\mathfrak{i})}(0) + \int_0^T \mathfrak{f}''(\eta)\mathfrak{g}''(\eta)d\eta, \tag{11} $$

and norm:

$$ \|\mathfrak{f}\|_{\mathscr{W}_2^2} = \sqrt{\langle \mathfrak{f}, \mathfrak{f}\rangle_{\mathscr{W}_2^2}}, $$

for all $\mathfrak{f}, \mathfrak{g} \in \mathscr{W}_2^2(\mathscr{I})$.

Theorem 3.1. The RK function $\mathscr{S}_\tau(\eta)$ is associated with the space $\mathscr{W}_2^2(\mathscr{I})$, given by

$$ \mathscr{S}_\tau(\eta) = \begin{cases} \eta\tau + \frac{1}{2}\eta^2\tau - \frac{1}{6}\eta^3 & ,\eta \leqslant \tau, \\ \tau\eta + \frac{1}{2}\tau^2\eta - \frac{1}{6}\tau^3 & ,\eta > \tau. \end{cases} \tag{12} $$

Proof. By Definition 3.2, we have

$$ \begin{aligned} \langle \mathfrak{f}, \mathscr{S}_\tau\rangle_{\mathscr{W}_2^2} &= \mathfrak{f}(0)\mathscr{S}_\tau(0) + \mathfrak{f}'(0)\mathscr{S}'_\tau(0) + \textstyle\int_0^T \mathfrak{f}''(\eta)\mathscr{S}''_\tau(\eta)d\eta \\ &= \mathfrak{f}(\tau) \end{aligned} \tag{13} $$

Integrating by parts, we get

$$ \begin{aligned} \langle \mathfrak{f}, \mathscr{S}_\tau\rangle_{\mathscr{W}_2^2} = {}& \mathfrak{f}(0)\mathscr{S}_\tau(0) + \mathfrak{f}'(0)\mathscr{S}'_\tau(0) + \mathfrak{f}'(T)\mathscr{S}''_\tau(T) \\ & -\mathfrak{f}'(0)\mathscr{S}''_\tau(0) - \int_0^T \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_\tau(\eta)d\eta. \end{aligned} \tag{14} $$

By using integration by parts on more time, yields

$$ \begin{aligned} \langle \mathfrak{f}, \mathscr{S}_\tau\rangle_{\mathscr{W}_2^2} = {}& \mathfrak{f}(0)\mathscr{S}_\tau(0) + \mathfrak{f}'(0)\mathscr{S}'_\tau(0) + \mathfrak{f}'(T)\mathscr{S}''_\tau(T) \\ & -\mathfrak{f}'(0)\mathscr{S}''_\tau(0) - \mathfrak{f}(T)\mathscr{S}^{(3)}_\tau(T) \\ & +\mathfrak{f}(0)\mathscr{S}^{(3)}_\tau(0) + \textstyle\int_0^T \mathfrak{f}(\eta)\mathscr{S}^{(4)}_\tau(\eta)d\eta. \end{aligned} \tag{15} $$

Taking

$$ \begin{aligned} & \mathscr{S}'_\tau(0) - \mathscr{S}''_\tau(0) = 0, \\ & \mathscr{S}_\tau(0) = 0, \\ & \mathscr{S}''_\tau(T) = 0, \\ & \mathscr{S}^{(3)}_\tau(T) = 0, \end{aligned} \tag{16} $$

then (13) gives

$$\mathscr{S}_{\tau}^{(4)}(\eta) = \delta(\eta - \tau), \quad (17)$$

where δ is called the Dirac-Delta function.

For $\eta \neq \tau$, (17) gives

$$\mathscr{S}_{\tau}^{(4)}(\eta) = 0. \quad (18)$$

As a result,

$$\mathscr{S}_{\tau}(\eta) = \begin{cases} \mathfrak{h}_1(\tau) + \mathfrak{h}_2(\tau)\eta + \mathfrak{h}_3(\tau)\eta^2 + \mathfrak{h}_4(\tau)\eta^3 & , \eta \leqslant \tau, \\ \mathfrak{k}_1(\tau) + \mathfrak{k}_2(\tau)\eta + \mathfrak{k}_3(\tau)\eta^2 + \mathfrak{k}_4(\tau)\eta^3 & , \eta > \tau, \end{cases} \quad (19)$$

where $\mathfrak{h}_{\mathfrak{i}}(\tau)$ and $\mathfrak{k}_{\mathfrak{i}}(\tau), \mathfrak{i} = 1, \ldots, 4$ are unknown coefficients that we will calculate below.

Then the Dirac-Delta function property implies

$$\mathscr{S}_{\eta^+}(\eta) = \mathscr{S}_{\eta^-}(\eta), \quad (20)$$
$$\mathscr{S}'_{\eta^+}(\eta) = \mathscr{S}'_{\eta^-}(\eta), \quad (21)$$
$$\mathscr{S}''_{\eta^+}(\eta) = \mathscr{S}''_{\eta^-}(\eta), \quad (22)$$
$$\mathscr{S}^{(3)}_{\eta^+}(\eta) - \mathscr{S}^{(3)}_{\eta^-}(\eta) = 1. \quad (23)$$

By (16), (20)–(23), (19), and the derivatives of (19), we can, then, obtain the unknown coefficients as

$$\mathfrak{h}_1(\tau) = 0, \mathfrak{k}_1(\tau) = -\tfrac{\tau^3}{6},$$
$$\mathfrak{h}_2(\tau) = \tau, \mathfrak{k}_2(\tau) = \tfrac{1}{2}\tau(2+\tau),$$
$$\mathfrak{h}_3(\tau) = \tfrac{\tau}{2}, \mathfrak{k}_3(\tau) = 0,$$
$$\mathfrak{h}_4(\tau) = -\tfrac{1}{6}, \mathfrak{k}_4(\tau) = 0.$$

□

Theorem 3.2. The RK function $\mathscr{S}_{\eta}(\eta)$ of the space $\mathscr{W}_2^2(\mathscr{I})$ is obtained as:

$$\mathscr{S}_{\tau}(\eta) = \begin{cases} \eta\tau + \frac{1}{2}\eta^2\tau - \frac{1}{6}\eta^3 & , \eta \leqslant \tau, \\ \tau\eta + \frac{1}{2}\tau^2\eta - \frac{1}{6}\tau^3 & , \eta > \tau. \end{cases} \quad (24)$$

Proof. It suffices to show that

$$\langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} = \mathfrak{f}(\tau).$$

We have

$$\langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} = \mathfrak{f}(0)\mathscr{S}_{\tau}(0) + \mathfrak{f}'(0)\mathscr{S}'_{\tau}(0) + \int_0^T \mathfrak{f}''(\tau)\mathscr{S}''_{\tau}(\eta)d\eta.$$

The use of integration by parts gives

$$\langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} = \mathfrak{f}(0)\mathscr{S}_{\tau}(0) + \mathfrak{f}'(0)\mathscr{S}'_{\tau}(0) + \mathfrak{f}'(T)\mathscr{S}''_{\tau}(T) - \mathfrak{f}'(0)\mathscr{S}''_{\tau}(0) - \int_0^T \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_{\tau}(\eta)d\eta. \quad (25)$$

Since $\mathscr{S}_{\tau}(\eta) \in \mathscr{W}_2^2(\mathscr{I})$, we have

$$\mathscr{S}_{\tau}(0) = 0. \quad (26)$$

Then we get

$$\langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} = \mathfrak{f}'(0)(\mathscr{S}'_{\tau}(0) - \mathscr{S}''_{\tau}(0)) + \mathfrak{f}'(T)\mathscr{S}''_{\tau}(T) - \int_0^T \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_{\tau}(\eta)d\eta. \quad (27)$$

What we need now is to calculate $\mathscr{S}'_{\tau}(0), \mathscr{S}''_{\tau}(0)$, and $\mathscr{S}''_{\tau}(T)$. So,

$$\mathscr{S}'_{\tau}(\eta) = \begin{cases} \tau + \eta\tau - \frac{1}{2}\eta^2 & , \eta \leqslant \tau, \\ \tau + \frac{1}{2}\tau^2 & , \eta > \tau. \end{cases}$$

$$\mathscr{S}''_{\tau}(\eta) = \begin{cases} \tau - \eta & , \eta \leqslant \tau, \\ 0 & , \eta > \tau. \end{cases}$$

Therefore, we obtain

$$\mathscr{S}'_{\tau}(0) = \tau, \qquad \mathscr{S}''_{\tau}(0) = \tau, \qquad \mathscr{S}''_{\tau}(T) = 0.$$

Now by using the above results, we can simplify (27) to

$$\langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} = -\int_0^T \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_{\tau}(\eta)d\eta. \quad (28)$$

We have

$$\mathscr{S}^{(3)}_{\tau}(\eta) = \begin{cases} -1 & , \eta \leqslant \tau, \\ 0 & , \eta > \tau. \end{cases}$$

Then, we get

$$\begin{aligned} \langle \mathfrak{f}, \mathscr{S}_{\tau} \rangle_{\mathscr{W}_2^2} &= -\int_0^{\tau} \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_{\tau}(\eta)d\eta - \int_{\tau}^{T} \mathfrak{f}'(\eta)\mathscr{S}^{(3)}_{\tau}(\eta)d\eta \\ &= \int_0^{\tau} \mathfrak{f}'(\eta)d\eta \\ &= \mathfrak{f}(\tau) - \mathfrak{f}(0). \end{aligned}$$

As $\mathfrak{f}(0) = 0$ because $\mathfrak{f}(\eta) \in \mathscr{W}_2^2(\mathscr{I})$, we conclude the desired result

$$\langle \mathfrak{f}, \mathscr{S}_{\eta} \rangle_{\mathscr{W}_2^2} = \mathfrak{f}(\eta).$$

□

Definition 3.3. Given an interval $\mathscr{I} \subset \mathbb{R}$ and $\mathfrak{f} : \mathscr{I} \to \mathbb{R}$. A function space $\mathscr{W}_2^1(\mathscr{I})$ is [29]

$$\mathscr{W}_2^1(\mathscr{I}) = \{\mathfrak{f}(\eta) | \mathfrak{f} \text{ is an a.c. function on } \mathscr{I}, \mathfrak{f}' \in L^2(\mathscr{I})\},$$

with the inner product:

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathscr{W}_2^1} = \mathfrak{f}(0)\mathfrak{g}(0) + \int_0^T \mathfrak{f}'(\eta)\mathfrak{g}'(\eta)d\eta, \quad (29)$$

and norm:

$$\|\mathfrak{f}\|_{\mathscr{W}_2^1} = \sqrt{\langle \mathfrak{f}, \mathfrak{f} \rangle_{\mathscr{W}_2^1}}, \quad (30)$$

for all $\mathfrak{f}, \mathfrak{g} \in \mathscr{W}_2^1(\mathscr{I})$.

Theorem 3.3. The RK function $\mathscr{R}_{\eta}(\eta)$ is associated with the space $\mathscr{W}_2^1(\mathscr{I})$, given by [29]

$$\mathscr{R}_{\tau}(\eta) = \begin{cases} 1 + \eta & , \eta \leqslant \tau, \\ 1 + \tau & , \eta > \tau. \end{cases} \quad (31)$$

Proof. By Definition 3.3, we have

$$\langle \mathfrak{f}, \mathscr{R}_{\tau} \rangle_{\mathscr{W}_2^1} = \mathfrak{f}(0)\mathscr{R}_{\tau}(0) + \int_0^T \mathfrak{f}'(\eta)\mathscr{R}'_{\tau}(\eta)d\eta = \mathfrak{f}(\tau). \quad (32)$$

Integrating by parts, we get

$$\langle \mathfrak{f}, \mathscr{R}_\tau \rangle_{\mathscr{W}_2^1} = \mathfrak{f}(0)\mathscr{R}_\tau(0) + \mathfrak{f}(T)\mathscr{R}'_\tau(T) - \mathfrak{f}(0)\mathscr{R}'_\tau(0) - \int_0^T \mathfrak{f}(\eta)\mathscr{R}''_\tau(\eta)d\eta. \tag{33}$$

Taking

$$\mathscr{R}_\tau(0) - \mathscr{R}'_\tau(0) = 0, \quad \mathscr{R}'_\tau(T) = 0, \tag{34}$$

then (32) gives

$$-\mathscr{R}''_\tau(\eta) = \delta(\eta - \tau), \tag{35}$$

where δ is called the Dirac-Delta function.

For $\eta \neq \tau$, (35) gives

$$\mathscr{R}''_\tau(\eta) = 0. \tag{36}$$

As a result,

$$\mathscr{R}_\eta(\eta) = \begin{cases} \mathfrak{h}_1(\tau) + \mathfrak{h}_2(\tau)\eta & ,\eta \leqslant \tau, \\ \mathfrak{k}_1(\tau) + \mathfrak{k}_2(\tau)\eta & ,\eta > \tau, \end{cases} \tag{37}$$

where $\mathfrak{h}_\mathfrak{i}(\tau)$ and $\mathfrak{k}_\mathfrak{i}(\tau), \mathfrak{i} = 1,2$ are unknown coefficients that we will calculate below.

Then the Dirac-Delta function property implies

$$\mathscr{R}_{\eta^+}(\eta) = \mathscr{R}_{\eta^-}(\eta), \tag{38}$$

$$\mathscr{R}'_{\eta^+}(\eta) - \mathscr{R}'_{\eta^-}(\eta) = -1. \tag{39}$$

By (34), (38), (39), (37), and the first derivative of (37), we can, then, obtain the unknown coefficients as

$$\mathfrak{h}_1(\tau) = 1, \quad \mathfrak{k}_1(\tau) = 1 + \tau,$$
$$\mathfrak{h}_2(\tau) = 1, \quad \mathfrak{k}_2(\tau) = 0.$$

□

4. The RKHS approach

Let us consider the following fractal fractional differential equation (FFDE) with the Mittag–Leffler kernel as

$${}^{0}_{FFM}D^{\eta}_{\gamma,\beta}\mathfrak{f}(\eta) = \phi(\eta, \mathfrak{f}(\eta)), \qquad \eta \in \mathscr{I}, \quad 0 < \gamma, \beta \leq 1. \tag{40}$$

Here ${}^{0}_{FFM}D^{\eta}_{\gamma,\beta}\mathfrak{f}(\eta)$ is given in Definition 2.3. Briefly, this operator denotes the fractal-fractional operator of fractal order β and fractional order γ with respect to η. $\mathfrak{f}$ is an unknown function to be obtained. Further, $\phi(\eta, \mathfrak{f}(\eta))$ is an inhomogeneous term.

Noting that by converting problem (40) into Atangana-Baleanu fractional differential equation, the RKHSM will become simple and easy. So, let us apply the following steps:

By Definition 2.3, we have

$$\frac{AB(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\eta^\beta}\int_0^\eta \mathfrak{f}(\zeta)E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^\gamma\right)\mathrm{d}\zeta = \phi(\eta, \mathfrak{f}(\eta)). \tag{41}$$

Then, we obtain

$$\frac{AB(\gamma)}{1-\gamma}\frac{1}{\beta\eta^{\beta-1}}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^\eta \mathfrak{f}(\zeta)E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^\gamma\right)\mathrm{d}\zeta = \phi(\eta, \mathfrak{f}(\eta)). \tag{42}$$

Therefore, we get

$${}^{0}_{ABR}D^{\eta}_{\gamma,\beta}(\eta) = \beta\eta^{\beta-1}\phi(\eta, \mathfrak{f}(\eta)). \tag{43}$$

Thanks to (10), we find

$${}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \beta\eta^{\beta-1}\phi(\eta, \mathfrak{f}(\eta)) - \frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)E_\gamma\left(-\frac{\gamma}{1-\gamma}\eta^\gamma\right). \tag{44}$$

Then, (40) becomes

$$\begin{cases} {}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \Lambda(\eta, \mathfrak{f}(\eta)), \quad \eta \in \mathscr{I}, \quad 0 < \gamma < 1, \\ \mathfrak{f}(0) = 0, \end{cases} \tag{45}$$

where $\Lambda(\eta, \mathfrak{f}(\eta)) = \beta\eta^{\beta-1}\phi(\eta, \mathfrak{f}(\eta))$.

According to the RKHSM, we apply the operator ${}^{0}_{AB}I^{\eta}_{\gamma}$ on both sides of (45) and obtain:

$$\begin{cases} \mathfrak{f}(\eta) = \Theta(\eta, \mathfrak{f}(\eta)), \quad \eta \in \mathscr{I}, \quad 0 < \gamma < 1, \\ \mathfrak{f}(0) = 0, \end{cases} \tag{46}$$

where $\Theta(\eta, \mathfrak{f}(\eta)) = {}^{0}_{AB}I^{\eta}_{\gamma}(\Lambda(\eta, \mathfrak{f}(\eta))) = {}^{0}_{AB}I^{\eta}_{\gamma}(\beta\eta^{\beta-1}\phi(\eta, \mathfrak{f}(\eta)))$.

We next introduce a linear operator $\mathfrak{O} : \mathscr{W}_2^2(\mathscr{I}) \to \mathscr{W}_2^1(\mathscr{I})$ defined by

$$\mathfrak{O}\mathfrak{f}(\eta) = \mathfrak{f}(\eta) \tag{47}$$

to obtain the solution's representation of (46).

Now by applying this latter, (46) can be rewritten as follows

$$\begin{cases} \mathfrak{O}\mathfrak{f}(\eta) = \Theta(\eta, \mathfrak{f}(\eta)), \quad \eta \in \mathscr{I}, \quad 0 < \gamma < 1, \\ \mathfrak{f}(0) = 0, \end{cases} \tag{48}$$

where $\Theta(\eta, \mathfrak{f}(\eta)) = {}^{0}_{AB}I^{\eta}_{\gamma}(\Lambda(\eta, \mathfrak{f}(\eta))) = {}^{0}_{AB}I^{\eta}_{\gamma}(\beta\eta^{\beta-1}\phi(\eta, \mathfrak{f}(\eta)))$.

One of the essential steps for applying the RKHSM is the construction of an orthogonal function system of $\mathscr{W}_2^2(\mathscr{I})$. To do so, let

$$\psi_\mathfrak{i}(\eta) = \mathfrak{O}^*\kappa_\mathfrak{i}(\eta),$$

where

- $\kappa_\mathfrak{i}(\eta) = \mathscr{R}_{\eta_\mathfrak{i}}(\eta)$, in which $\mathscr{R}_{\eta_\mathfrak{i}}(\eta)$ represents the RKF of the space $\mathscr{W}_2^1(\mathscr{I})$.
- The countable set $\{\eta_\mathfrak{i}\}_{\mathfrak{i}=1}^\infty$ is dense in $\mathscr{I}$.
- $\mathfrak{O}^*$ denotes the formal adjoint of $\mathfrak{O}$.

Based on Gram-Schmidt's process, the orthonormal system $\{\bar{\psi}_\mathfrak{i}\}_{\mathfrak{i}=1}^\infty$ in $\mathscr{W}_2^2(\mathscr{I})$ is found as:

$$\bar{\psi}_\mathfrak{i}(\eta) = \sum_{k=1}^{\mathfrak{i}} \varpi_{\mathfrak{i}k}\psi_k(\eta), \quad \varpi_{\mathfrak{i}\mathfrak{i}} > 0, \ \mathfrak{i} = 1, 2, \ldots. \tag{49}$$

Where $\{\psi_\mathfrak{i}\}_{\mathfrak{i}=1}^\infty$ denotes the function system in $\mathscr{W}_2^2(\mathscr{I})$ that can be determined in the following way

$$\psi_\mathfrak{i}(\eta) = \mathfrak{O}^*\kappa_\mathfrak{i}(\eta) = \langle \mathfrak{O}^*\kappa_\mathfrak{i}(\eta), \mathscr{S}_\eta(\eta)\rangle_{\mathscr{W}_2^2} = \langle \kappa_\mathfrak{i}(\eta), \ \mathfrak{O}\mathscr{S}_\eta(\eta)\rangle_{\mathscr{W}_2^1} = \langle \mathscr{R}_{\eta_\mathfrak{i}}(\eta), \mathfrak{O}\mathscr{S}_\eta(\eta)\rangle_{\mathscr{W}_2^1} = \mathfrak{O}_\eta\mathscr{S}_\eta(\eta)|_{\eta=\eta_\mathfrak{i}},$$

the operator $\mathfrak{O}_\eta$ is introduced to tell that $\mathfrak{O}$ is applied to η.

And, the coefficients $\varpi_{\mathfrak{i}k}$ are the orthogonalization which take the following values:

$$\varpi_{\mathfrak{i}\mathfrak{j}} = \begin{cases} \frac{1}{\|\psi_1\|}, & \text{for} \quad \mathfrak{i} = \mathfrak{j} = 1, \\ \frac{1}{e_\mathfrak{i}}, & \text{for} \quad \mathfrak{i} = \mathfrak{j} \neq 1, \\ -\frac{1}{e_\mathfrak{i}}\sum_{k=\mathfrak{j}}^{\mathfrak{i}-1} C_{\mathfrak{i}k}\varpi_{k\mathfrak{j}}, & \text{for} \quad \mathfrak{i} > \mathfrak{j}, \end{cases} \tag{50}$$

where $e_{\mathfrak{i}}=\sqrt{\|\psi_{\mathfrak{i}}\|^2-\sum_{k=1}^{\mathfrak{i}-1}C_{\mathfrak{i}k}^2}$, $C_{\mathfrak{i}k}=\langle\psi_{\mathfrak{i}},\bar{\psi}_k\rangle_{\mathscr{W}_2^2}$.

Theorem 4.1. Suppose $\{\eta_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is a dense set on $\mathscr{I}$, then $\{\psi_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is the complete system of the space $\mathscr{W}_2^2(\mathscr{I})$.

Proof. It is clear that $\psi_{\mathfrak{i}}(\eta)\in\mathscr{W}_2^2(\mathscr{I})$, and so for each fixed $\mathfrak{f}(\eta)\in\mathscr{W}_2^2(\mathscr{I})$, we put

$$\langle\mathfrak{f}(\eta),\psi_{\mathfrak{i}}(\eta)\rangle_{\mathscr{W}_2^2}=0,\ \mathfrak{i}=1,2,\ldots.$$

Since

$$\begin{aligned}\langle\mathfrak{f}(\eta),\psi_{\mathfrak{i}}(\eta)\rangle_{\mathscr{W}_2^2}&=\langle\mathfrak{f}(\eta),\mathfrak{D}^*\kappa_{\mathfrak{i}}(\eta)\rangle_{\mathscr{W}_2^2}=\langle\mathfrak{D}\mathfrak{f}(\eta),\kappa_{\mathfrak{i}}(\eta)\rangle_{\mathscr{W}_2^1}\\&=\mathfrak{D}\mathfrak{f}(\eta_{\mathfrak{i}})=0,\end{aligned}$$

and $\{\eta_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is dense on the interval $\mathscr{I}$, we have

$$\mathfrak{D}\mathfrak{f}(\eta)=0.$$

By applying the operator $\mathfrak{D}^{-1}$ to the last result, we reach

$$\mathfrak{f}(\eta)=0.$$

□

Lemma 4.1. Assume $\mathfrak{f}(\eta)\in\mathscr{W}_2^2(\mathscr{I})$, then

$$\|\mathfrak{f}^{(\mathfrak{i})}(\eta)\|_C\leqslant\mathfrak{C}\|\mathfrak{f}(\eta)\|_{\mathscr{W}_2^2},\quad\mathfrak{i}=0,1,$$

where $\mathfrak{C}$ is a non-negative constant and $\|\mathfrak{f}(\eta)\|_C=\max_{\eta\in\mathscr{I}}|\mathfrak{f}(\eta)|$.

Proof. $\forall\eta\in\mathscr{I}$ we have

$$\mathfrak{f}^{(\mathfrak{i})}(\eta)=\left\langle\mathfrak{f}(\cdot),\partial_\eta^{(\mathfrak{i})}\mathscr{S}_\eta(\cdot)\right\rangle_{\mathscr{W}_2^2},\quad\mathfrak{i}=0,1.$$

Using the expression of $\partial_\eta^{(\mathfrak{i})}\mathscr{S}_\eta(\cdot)$, we can reach

$$\left\|\partial_\eta^{(\mathfrak{i})}\mathscr{S}_\eta\right\|_{\mathscr{W}_2^2}\leqslant\mathfrak{C}_{\mathfrak{i}},\quad\mathfrak{i}=0,1.$$

Consequently,

$$\begin{aligned}\left|\mathfrak{f}^{(\mathfrak{i})}(\eta)\right|&=\left|\left\langle\mathfrak{f}(\cdot),\partial_\eta^{(\mathfrak{i})}\mathscr{S}_\eta(\cdot)\right\rangle_{\mathscr{W}_2^2}\right|\leqslant\left\|\partial_\eta^{(\mathfrak{i})}\mathscr{S}_\eta\right\|_{\mathscr{W}_2^2}\|\mathfrak{f}\|_{\mathscr{W}_2^2}\\&\leqslant\mathfrak{C}_{\mathfrak{i}}\|\mathfrak{f}\|_{\mathscr{W}_2^2},\ \mathfrak{i}=0,1.\end{aligned}\tag{51}$$

where $\mathfrak{C}=\max_{\mathfrak{i}=0,1}\{\mathfrak{C}_{\mathfrak{i}}\}$. Then Lemma 4.1 follows from (51). □

Theorem 4.2. Assume $\{\eta_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\infty}$ is a dense set on $\mathscr{I}$ and the solution of (48) is unique on $\mathscr{W}_2^2(\mathscr{I})$. Therefore, the solution of (48) is

$$\mathfrak{f}(\eta)=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\Theta(\eta_k,\mathfrak{f}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta).\tag{52}$$

Proof. Since $\{\bar{\psi}_{\mathfrak{i}}(\eta)\}_{\mathfrak{i}=1}^{\infty}$ is a complete orthonormal basis in $\mathscr{W}_2^2(\mathscr{I})$, we have

$$\begin{aligned}\mathfrak{f}(\eta)&=\sum_{\mathfrak{i}=1}^{\infty}\langle\mathfrak{f}(\eta),\bar{\psi}_{\mathfrak{i}}(\eta)\rangle_{\mathscr{W}_2^2}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\left\langle\mathfrak{f}(\eta),\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\psi_k(\eta)\right\rangle_{\mathscr{W}_2^2}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\langle\mathfrak{f}(\eta),\psi_k(\eta)\rangle_{\mathscr{W}_2^2}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\langle\mathfrak{f}(\eta),\mathfrak{D}^*\kappa_k(\eta)\rangle_{\mathscr{W}_2^2}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\langle\mathfrak{D}\mathfrak{f}(\eta),\kappa_k(\eta)\rangle_{\mathscr{W}_2^1}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\langle\mathfrak{D}\mathfrak{f}(\eta),\mathscr{R}_\eta(\eta_k)\rangle_{\mathscr{W}_2^1}\bar{\psi}_{\mathfrak{i}}(\eta)\\&=\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\Theta(\eta_k,\mathfrak{f}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta).\end{aligned}$$

with $\Theta(\eta_k,\mathfrak{f}(\eta_k))=\mathfrak{D}\mathfrak{f}(\eta_k)$. □

We now write the approximate RKHSM solution $\mathfrak{f}_n(\eta)$ as

$$\mathfrak{f}_n(\eta)=\sum_{\mathfrak{i}=1}^{n}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\Theta(\eta_k,\mathfrak{f}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta).\tag{53}$$

The space $\mathscr{W}_2^2(\mathscr{I})$ is a Hilbert space, so this latter implies that

$$\sum_{\mathfrak{i}=1}^{\infty}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\Theta(\eta_k,\mathfrak{f}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta)<\infty,$$

we conclude, then, that $\mathfrak{f}_n(\eta)$ converges to $\mathfrak{f}(\eta)$ in the norm.

Theorem 4.3. The approximate solution $\mathfrak{f}_n(\eta)$ and $\mathfrak{f}_n'(\eta)$ converge uniformly to the exact solution $\mathfrak{f}(\eta)$ and $\mathfrak{f}'(\eta)$, respectively.

Proof. Firstly, to demonstrate that $\mathfrak{f}_n(\eta)$ is uniformly convergent to $\mathfrak{f}_n(\eta)$, we estimate the term on the left below: $\forall\eta\in\mathscr{I}$,

$$\begin{aligned}|\mathfrak{f}_n(\eta)-\mathfrak{f}(\eta)|&=\left|\left\langle\mathfrak{f}_n(\cdot)-\mathfrak{f}(\cdot),\mathscr{S}_\eta(\cdot)\right\rangle_{\mathscr{W}_2^2}\right|\\&\leqslant\left\|\mathscr{S}_\eta\right\|_{\mathscr{W}_2^2}\|\mathfrak{f}_n-\mathfrak{f}\|_{\mathscr{W}_2^2}\\&\leqslant\mathscr{C}_0\|\mathfrak{f}_n-\mathfrak{f}\|_{\mathscr{W}_2^2},\end{aligned}$$

where $\mathscr{C}_0$ is a constant.

Following the same way, we get

$$|\mathfrak{f}_n'(\eta)-\mathfrak{f}'(\eta)|\leqslant\left\|\partial_\eta\mathscr{S}_\eta\right\|_{\mathscr{W}_2^2}\left\|\mathfrak{f}_n'-\mathfrak{f}'\right\|_{\mathscr{W}_2^2},$$

due to the uniform boundedness of $\partial_\eta\mathscr{S}_\eta(\cdot)$, we have

$$\left\|\partial_\eta\mathscr{S}_\eta\right\|_{\mathscr{W}_2^2}\leqslant\mathscr{C}_1,$$

where $\mathscr{C}_1$ is a positive constant.

Therefore

$$|\mathfrak{f}_n'(\eta)-\mathfrak{f}'(\eta)|\leqslant\mathscr{C}_1\left\|\mathfrak{f}_n'-\mathfrak{f}'\right\|_{\mathscr{W}_2^2}.$$

□

5. Efficient implementation

The n-term approximate solution of problem (52) obtained via RKHS method is given by

$$\mathfrak{f}_n(\eta) = \sum_{\mathrm{i}=1}^{n} \varkappa_{\mathrm{i}} \bar{\psi}_{\mathrm{i}}(\eta). \tag{54}$$

where

$$\varkappa_{\mathrm{i}} = \sum_{k=1}^{\mathrm{i}} \varpi_{\mathrm{i}k} \Theta(\eta_k, \mathfrak{f}(\eta_k)). \tag{55}$$

Here, by letting $\eta_1 = 0$, the values of $\mathfrak{f}(\eta_1)$ will be known from the ICs. And, $\mathfrak{f}_0(\eta_1) = \mathfrak{f}(\eta_1)$.

Theorem 5.1. *Assume $\|\mathfrak{f}_n\|_{\mathscr{W}_2^2}$ is bounded in (48), let $\{\eta_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ be a dense set on $\mathscr{I}$, and (54) has at most one solution $\mathfrak{f}(\eta)$. Then*

1. $\mathfrak{f}_n(\eta)$ converges to $\tilde{\mathfrak{f}}(\eta)$
2.

$$\mathfrak{f}_n(\eta) = \sum_{\mathrm{i}=1}^{n} \varkappa_{\mathrm{i}} \bar{\psi}_{\mathrm{i}}(\eta),$$

where $\varkappa_{\mathrm{i}}$ is given by (55).

Proof.

1. We derive from (54) that

$$\mathfrak{f}_{n+1}(\eta) = \mathfrak{f}_n(\eta) + \varkappa_{n+1} \bar{\psi}_{n+1}(\eta),$$

the orthogonality of $\{\bar{\psi}_{\mathrm{i}}(\eta)\}_{\mathrm{i}=1}^{\infty}$ implies

$$\begin{aligned} \|\mathfrak{f}_{n+1}\|_{\mathscr{W}_2^2}^2 &= \|\mathfrak{f}_n\|_{\mathscr{W}_2^2}^2 + \varkappa_{n+1}^2 \\ &= \|\mathfrak{f}_{n-1}\|_{\mathscr{W}_2^2}^2 + \varkappa_n^2 + \varkappa_{n+1}^2 \\ &\vdots \\ &= \|\mathfrak{f}_0\|_{\mathscr{W}_2^2}^2 + \sum_{\mathrm{i}=1}^{n+1} \varkappa_{\mathrm{i}}^2, \end{aligned} \tag{56}$$

and so

$$\|\mathfrak{f}_n\|_{\mathscr{W}_2^2} \leqslant \|\mathfrak{f}_{n+1}\|_{\mathscr{W}_2^2}.$$

The convergence of $\|\mathfrak{f}_n\|_{\mathscr{W}_2^2}$ follows easily from the fact that $\|\mathfrak{f}_n\|_{\mathscr{W}_2^2}$ is bounded. Thus, it exists a positive constant σ such that

$$\sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}}^2 = \sigma.$$

Conseauently

$$\{\sigma_{\mathrm{i}}^2\}_{\mathrm{i}=1}^{\infty} \in \ell^2.$$

Since $\left(\mathfrak{f}_{\mathfrak{q}}(\eta) - \mathfrak{f}_{\mathfrak{q}-1}(\eta)\right) \perp \cdots \perp \left(\mathfrak{f}_{n+1}(\eta) - \mathfrak{f}_n(\eta)\right)$, for $\mathfrak{q} > n$ we have

$$\begin{aligned} \|\mathfrak{f}_{\mathfrak{q}} - \mathfrak{f}_n\|_{\mathscr{W}_2^2}^2 &= \|\mathfrak{f}_{\mathfrak{q}} - \mathfrak{f}_{\mathfrak{q}-1} + \mathfrak{f}_{\mathfrak{q}-1} - \cdots + \mathfrak{f}_{n+1} - \mathfrak{f}_n\|_{\mathscr{W}_2^2}^2 \\ &= \|\mathfrak{f}_{\mathfrak{q}} - \mathfrak{f}_{\mathfrak{q}-1}\|_{\mathscr{W}_2^2}^2 + \|\mathfrak{f}_{\mathfrak{q}-1} - \mathfrak{f}_{\mathfrak{q}-2}\|_{\mathscr{W}_2^2}^2 + \cdots \\ &\quad + \|\mathfrak{f}_{n+1} - \mathfrak{f}_n\|_{\mathscr{W}_2^2}^2. \end{aligned} \tag{57}$$

Furthermore,

$$\|\mathfrak{f}_{\mathfrak{q}} - \mathfrak{f}_{\mathfrak{q}-1}\|_{\mathscr{W}_2^2}^2 = \varkappa_q^2.$$

Consequently, we have

$$\|\mathfrak{f}_{\mathfrak{q}} - \mathfrak{f}_n\|_{\mathscr{W}_2^2}^2 = \sum_{\ell=n+1}^{\mathfrak{q}} \varkappa_\ell^2 \to 0, \quad \text{as} \quad n \to \infty.$$

On account of the completeness of $\mathscr{W}_2^2(\mathscr{I})$, we deduce that $\mathfrak{f}_n \to \tilde{\mathfrak{f}}$ as $n \to \infty$.

2. To demonstrate that $\tilde{\mathfrak{f}}(\eta)$ is the representation solution of (48). We must first take the limits in (54), to get

$$\tilde{\mathfrak{f}}(\eta) = \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \bar{\psi}_{\mathrm{i}}(\eta). \tag{58}$$

Applying the linear operator $\mathfrak{D}$ to (58), we find

$$\mathfrak{D}\tilde{\mathfrak{f}}(\eta) = \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \mathfrak{D} \bar{\psi}_{\mathrm{i}}(\eta),$$

it follows that

$$\begin{aligned} \mathfrak{D}\tilde{\mathfrak{f}}(\eta_\ell) &= \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \langle \mathfrak{D}\bar{\psi}_{\mathrm{i}}(\eta), \kappa_\ell(\eta) \rangle_{\mathscr{W}_2^1} \\ &= \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \langle \bar{\psi}_{\mathrm{i}}(\eta), \mathfrak{D}^* \kappa_\ell(\eta) \rangle_{\mathscr{W}_2^2} \\ &= \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \langle \bar{\psi}_{\mathrm{i}}(\eta), \psi_\ell(\eta) \rangle_{\mathscr{W}_2^2}. \end{aligned}$$

Multiplying the latter equality by $\varpi_{\mathrm{j}\ell}$ and taking the summation $\sum_{\ell=1}^{\mathrm{j}}$, we get

$$\begin{aligned} \sum_{\ell=1}^{\mathrm{j}} \varpi_{\mathrm{j}\ell} \mathfrak{D}\tilde{\mathfrak{f}}(\eta_\ell) &= \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \left\langle \bar{\psi}_{\mathrm{i}}(\eta), \sum_{\ell=1}^{\mathrm{j}} \varpi_{\mathrm{j}\ell} \psi_\ell(\eta) \right\rangle_{\mathscr{W}_2^2} \\ &= \sum_{\mathrm{i}=1}^{\infty} \varkappa_{\mathrm{i}} \langle \bar{\psi}_{\mathrm{i}}(\eta), \bar{\psi}_{\mathrm{j}}(\eta) \rangle_{\mathscr{W}_2^2} \\ &= \varkappa_{\mathrm{j}}. \end{aligned}$$

Thanks to (55), we find

$$\mathfrak{D}\tilde{\mathfrak{f}}(\eta_\ell) = \Theta(\eta_\ell, \mathfrak{f}(\eta_\ell)).$$

For all $t \in \mathscr{I}$, there exists a subsequence $\{\eta_{n\mathrm{j}}\}_{\mathrm{j}=1}^{\infty}$ such that

$$\eta_{n\mathrm{j}} \to t, \quad \text{as} \quad \mathrm{j} \to \infty.$$

This latter follows from the density of $\{\eta_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ on $\mathscr{I}$.On the other hand, it is well-known to us

$$\mathfrak{D}\tilde{\mathfrak{f}}(\eta_{n\mathrm{j}}) = \Theta(\eta_{n\mathrm{j}}, \mathfrak{f}(\eta_{n\mathrm{j}})).$$

Let $\mathrm{j} \to \infty$, we utilize the continuity of Θ to deduce

$$\mathfrak{D}\tilde{\mathfrak{f}}(t) = \Theta(t, \mathfrak{f}(t)).$$

We obtain therefore the desired result for $\mathfrak{f}(\eta)$.

□

6. Numerical Experiments

In this section, we show the simplicity, effectiveness, performance, and applicability of the proposed RKHSM in solving some interesting ordinary differential equations under the FF derivative in the RL sense with the generalized M-L kernel. Let's first present the procedure that allows us to get the numerical solutions using the RKHSM:

Put $\mathfrak{f}_n(\eta_1) = \mathfrak{f}(\eta_1)$, the n-term solution's representation of (40) takes the form:

$$\mathfrak{f}_n(\eta) = \sum_{\mathfrak{i}=1}^{n}\sum_{k=1}^{\mathfrak{i}} \varpi_{\mathfrak{i}k}\Theta(\eta_k, \mathfrak{f}_{k-1}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta). \tag{59}$$

Then,

$$\begin{aligned}
\mathfrak{f}_1(\eta) &= \varpi_{11}\Theta(\eta_1, \mathfrak{f}_0(\eta_1))\bar{\psi}_1(\eta),\\
\mathfrak{f}_2(\eta) &= \mathfrak{f}_1(\eta) + (\varpi_{21}\Theta(\eta_1, \mathfrak{f}_0(\eta_1)) + \varpi_{22}\Theta(\eta_2, \mathfrak{f}_1(\eta_2)))\bar{\psi}_2(\eta),\\
&\vdots\\
\mathfrak{f}_n(\eta) &= \mathfrak{f}_{n-1}(\eta) + \sum_{k=1}^{n}\varpi_{nk}\Theta(\eta_k, \mathfrak{f}_{k-1}(\eta_k))\bar{\psi}_n(\eta),
\end{aligned}$$

where $\eta_{\mathfrak{i}} = \frac{\mathfrak{i}}{n}$, $\mathfrak{i} = 1, 2, \ldots, n$, n is the number of iterations, $\Theta(\eta, \mathfrak{f}(\eta))$ is the right-hand side of (46), the orthogonalization coefficients $\varpi_{11}, \varpi_{21}, \varpi_{22}, \varpi_{31}$ etc can be obtained by using (50), and $\bar{\psi}_{\mathfrak{i}}(\eta)$ is given by (49).

Problem 1. We take into consideration the following FFDE with the M-L kernel:

$${}^{0}_{FFM}D^{\eta}_{\gamma,\beta}\mathfrak{f}(\eta) = \eta, \qquad 0 \leqslant \eta \leq 1, \quad 0 < \gamma, \beta \leq 1. \tag{60}$$

The exact solution to the above problem is [9]

$$\mathfrak{f}(\eta) = \frac{\beta\eta^{\beta}\Gamma(1+\beta)}{AB(\gamma)}\left[\frac{1-\gamma}{\Gamma(1+\beta)} + \frac{\gamma\eta^{\gamma}}{\Gamma(1+\gamma+\beta)}\right]. \tag{61}$$

We need first to convert problem (60) into FDE with ABC derivative before applying the RKHSM. Therefore, let us follow the same steps as used in Section 4:

We have

$$\frac{AB(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\eta^{\beta}}\int_0^{\eta}\mathfrak{f}(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta = \eta. \tag{62}$$

Then, we get

$$\frac{AB(\gamma)}{1-\gamma}\frac{1}{\beta\eta^{\beta-1}}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^{\eta}\mathfrak{f}(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta = \eta. \tag{63}$$

Therefore, we reach

Table 1 The ES, AS, AE, and RE of Problem 1 at $\gamma = 0.9$ and $\beta = 0.5$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0303467525	0.030346752	1.9×10^{-10}	$6.260966476 \times 10^{-9}$
0.2	0.0595317426	0.059531742	2.0×10^{-10}	$3.359552253 \times 10^{-9}$
0.3	0.0922204234	0.092220423	9.0×10^{-11}	$9.759226502 \times 10^{-10}$
0.4	0.1280377235	0.128037723	6.0×10^{-10}	$4.686118931 \times 10^{-9}$
0.5	0.1666435062	0.166643505	1.2×10^{-9}	$7.201000671 \times 10^{-9}$
0.6	0.2077709006	0.207770897	3.6×10^{-9}	$1.732677670 \times 10^{-8}$
0.7	0.2512087053	0.251208702	3.3×10^{-9}	$1.313648743 \times 10^{-8}$
0.8	0.2967859434	0.296785939	4.4×10^{-9}	$1.482549999 \times 10^{-8}$
0.9	0.3443611598	0.344361151	8.8×10^{-9}	$2.555456604 \times 10^{-8}$
1	0.3938151168	0.393815129	1.2×10^{-8}	$3.097900380 \times 10^{-8}$

Table 2 The ES, AS, AE, and RE of Problem 1 at $\gamma = 0.8$ and $\beta = 0.6$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0494710213	0.049471020	8.9×10^{-10}	$1.799033003 \times 10^{-8}$
0.2	0.0923885265	0.092388524	2.2×10^{-9}	$2.337952647 \times 10^{-8}$
0.3	0.1378168504	0.137816849	1.8×10^{-9}	$1.306081219 \times 10^{-8}$
0.4	0.1859655179	0.185965515	3.1×10^{-9}	$1.666975703 \times 10^{-8}$
0.5	0.2367181603	0.236718155	5.3×10^{-9}	$2.238949472 \times 10^{-8}$
0.6	0.2899184788	0.289918471	7.8×10^{-9}	$2.690411467 \times 10^{-8}$
0.7	0.3454176484	0.345417639	9.4×10^{-9}	$2.721343291 \times 10^{-8}$
0.8	0.4030828548	0.403082846	8.8×10^{-9}	$2.183173979 \times 10^{-8}$
0.9	0.4627972771	0.462797261	1.6×10^{-8}	$3.478845014 \times 10^{-8}$
1	0.5244581711	0.524458177	5.9×10^{-9}	$1.124970555 \times 10^{-8}$

$$ {}_{ABR}^{\ 0}D_{\gamma}^{\eta}\mathfrak{f}(\eta) = \beta\eta^{\beta}. \tag{64}$$

Thanks to (10), we find

$$ {}_{ABC}^{\ 0}D_{\gamma}^{\eta}\mathfrak{f}(\eta) = \beta\eta^{\beta} - \frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right). \tag{65}$$

Then, we obtain

$$ \begin{cases} {}_{ABC}^{\ 0}D_{\gamma}^{\eta}\mathfrak{f}(\eta) = \beta\eta^{\beta}, & 0 < \eta \le 1, \quad 0 < \gamma < 1, \\ \mathfrak{f}(0) = 0. \end{cases} \tag{66}$$

Several interesting results can be obtained by using the RKHSM in which $\eta_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$. Tables 1–5 show the exact and approximate solutions and both absolute and relative errors (ES(s), AS(s), AE, RE, for short respectively) of (60) at various $\eta \in [0,1]$ when $(\gamma;\beta) \in \{(0.1;1),(0.25;1),(0.5;0.77),(0.8;0.6),(0.9;0.5)\}$. Also, the comparison between the ESs and ASs of this problem, which are obtained by (61) and RKHSM respectively, is depicted in Fig. 1 for different values of γ and β. We can observe, from Fig. 1, that the graphs are in good agreement with each other and they are also almost much, which confirms the efficiency of the RKHSM. Moreover, one can conclude that the RKHSM's solution is stable and convergent.

Problem 2. We take into consideration the following FFDE with the Mittag–Leffler kernel:

$$ {}_{FFM}^{\ 0}D_{\gamma,\beta}^{\eta}\mathfrak{f}(\eta) = \eta^2, \qquad 0 \leqslant \eta \le 1, \quad 0 < \gamma, \beta \le 1. \tag{67}$$

Table 3 The ES, AS, AE, and RE of Problem 1 at $\gamma = 0.5$ and $\beta = 0.77$ using 10 points.

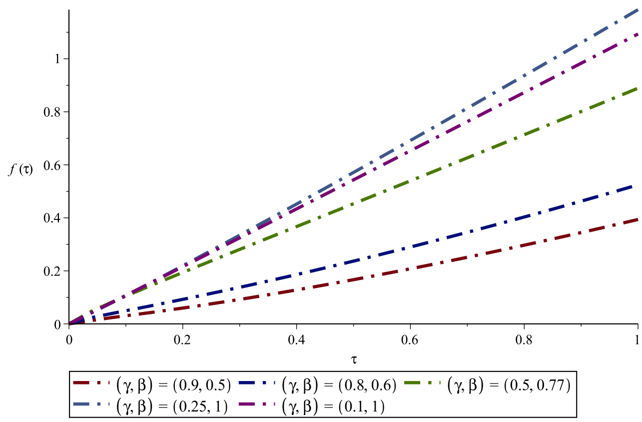
η	ES	AS	AE	RE
0.1	0.1049054230	0.1049054230	0	0
0.2	0.1939417178	0.1939417167	1.1×10^{-9}	$5.671807038 \times 10^{-9}$
0.3	0.2807892017	0.2807892009	8.0×10^{-10}	$2.849112413 \times 10^{-9}$
0.4	0.3670169696	0.3670169687	$9. \times 10^{-10}$	$2.452202690 \times 10^{-9}$
0.5	0.4531878467	0.4531878450	1.7×10^{-9}	$3.751203860 \times 10^{-9}$
0.6	0.5395608057	0.5395608063	$6. \times 10^{-10}$	$1.112015539 \times 10^{-9}$
0.7	0.6262701983	0.6262701990	$7. \times 10^{-10}$	$1.117728421 \times 10^{-5}$
0.8	0.7133901722	0.7133901723	$1. \times 10^{-10}$	$1.401757466 \times 10^{-10}$
0.9	0.8009626295	0.8009626279	1.6×10^{-9}	$1.997596319 \times 10^{-9}$
1	0.8890109501	0.8890109534	3.3×10^{-9}	$3.711990274 \times 10^{-9}$

Table 4 The ES, AS, AE, and RE of Problem 1 at $\gamma = 0.25$ and $\beta = 1$ using 10 points.

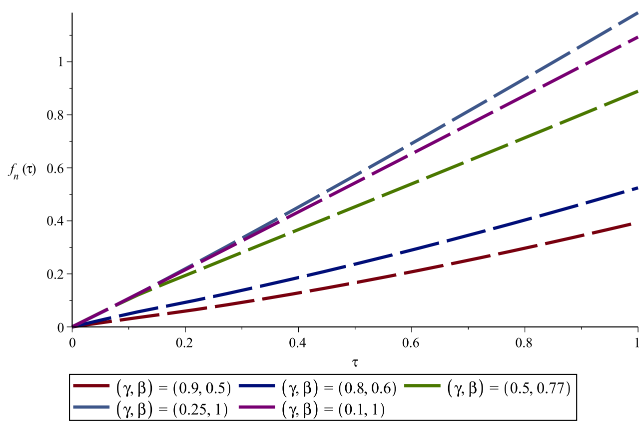
η	ES	AS	AE	RE
0.1	0.1067315291	0.1067315291	0	0
0.2	0.2191965198	0.2191965198	0	0
0.3	0.3345613689	0.3345613690	1.0×10^{-10}	$2.988988248 \times 10^{-10}$
0.4	0.4520295860	0.4520295863	3.0×10^{-10}	$6.636733729 \times 10^{-10}$
0.5	0.5711834965	0.5711834952	1.3×10^{-9}	$2.275976123 \times 10^{-9}$
0.6	0.6917597172	0.6917597166	6.0×10^{-10}	$8.673531937 \times 10^{-10}$
0.7	0.8135747809	0.8135747806	3.0×10^{-10}	$3.687429933 \times 10^{-10}$
0.8	0.9364925288	0.9364925276	1.2×10^{-9}	$1.281377014 \times 10^{-9}$
0.9	1.0604073510	1.0604073480	3.0×10^{-9}	$2.829101474 \times 10^{-9}$
1	1.1852346170	1.1852346220	5.0×10^{-9}	$4.218574051 \times 10^{-9}$

Table 5 The ES, AS, AE, and RE of Problem 1 at $\gamma = 0.1$ and $\beta = 1$ using 10 points.

η	ES	AS	AE	RE
0.1	0.1071820054	0.1071820055	1.0×10^{-10}	$9.329924331 \times 10^{-10}$
0.2	0.2155606830	0.2155606832	2.0×10^{-10}	$9.278129816 \times 10^{-10}$
0.3	0.3244501821	0.3244501820	1.0×10^{-10}	$3.082137275 \times 10^{-10}$
0.4	0.4336864888	0.4336864888	0	0
0.5	0.5431885395	0.5431885385	1.0×10^{-9}	$1.840981404 \times 10^{-9}$
0.6	0.6529072644	0.6529072650	6.0×10^{-10}	$9.189666477 \times 10^{-10}$
0.7	0.7628095468	0.7628095475	7.0×10^{-10}	$9.176602508 \times 10^{-10}$
0.8	0.8728714384	0.8728714389	5.0×10^{-10}	$5.728220423 \times 10^{-10}$
0.9	0.9830747673	0.9830747679	6.0×10^{-10}	$6.103299769 \times 10^{-10}$
1	1.0934052480	1.0934052490	1.0×10^{-9}	$9.145739897 \times 10^{-10}$



(a) ESs for different values of γ and β



(b) ASs for different values of γ and β

Fig. 1 Comparison between the exact and RKHSM's solutions for Problem 1 at various values of γ and β.

We need, on one hand, to convert problem (67) into FDE with ABC derivative which is given below by (72) to facilitate the RKHSM's application. On the other hand, we need to acquire the ES of (67). To do so, we can write (67) as:

$$\frac{AB(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\eta^{\beta}}\int_0^{\eta}\mathfrak{f}(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta=\eta^2. \tag{68}$$

We use the relation between the fractal and the classical derivatives and obtain

$$\frac{AB(\gamma)}{1-\gamma}\frac{1}{\beta\eta^{\beta-1}}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^{\eta}\mathfrak{f}(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta=\eta^2. \tag{69}$$

Therefore, we reach

$$ {}^{0}_{ABR}D^{\eta}_{\gamma}\mathfrak{f}(\eta)=\beta\eta^{\beta+1}. \tag{70}$$

We use the relation between RL and Caputo derivatives with the M-L kernel and get

$$ {}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta)=\beta\eta^{\beta+1}-\frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right). \tag{71}$$

Then, we obtain our first need as

$$\begin{cases} {}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta)=\beta\eta^{\beta+1}, \quad 0<\eta\le 1, \quad 0<\gamma<1, \\ \mathfrak{f}(0)=0. \end{cases} \tag{72}$$

Now, to get the exact solution, we implement the Laplace transform to the above equation and obtain:

$$\mathscr{L}\left({}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta)\right)=\mathscr{L}\left(\beta\eta^{\beta+1}\right) - \mathscr{L}\left(\frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right)\right). \tag{73}$$

Then, we have

$$\mathscr{L}\left(\frac{AB(\gamma)}{1-\gamma}\int_0^{\eta}\mathfrak{f}'(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta\right) =\beta\mathscr{L}\left(\eta^{\beta+1}\right)-\frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)\mathscr{L}\left(E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right)\right). \tag{74}$$

Then, we obtain

$$\frac{AB(\gamma)}{1-\gamma}\mathscr{L}\left(\mathfrak{f}'(\eta)*E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right)\right) =\beta\Gamma(2+\beta)s^{-2-\beta}-\frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)\frac{s^{\gamma-1}}{s^{\gamma}+\frac{\gamma}{1-\gamma}}. \tag{75}$$

Thanks to the convolution rule of Laplace transform, we find

$$\frac{AB(\gamma)}{1-\gamma}\mathscr{L}(\mathfrak{f}'(\eta))\mathscr{L}\left(E_{\gamma}\left(-\frac{\gamma}{1-\gamma}\eta^{\gamma}\right)\right) =\beta\Gamma(2+\beta)s^{-2-\beta}-\frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)\frac{s^{\gamma-1}}{s^{\gamma}+\frac{\gamma}{1-\gamma}}. \tag{76}$$

Which implies the following

$$\left(s\mathscr{L}(\mathfrak{f}(\eta))-\mathfrak{f}(0)\right)\frac{AB(\gamma)s^{\gamma-1}}{s^{\gamma}(1-\gamma)+\gamma} =\beta\Gamma(2+\beta)s^{-2-\beta}-\frac{AB(\gamma)\mathfrak{f}(0)s^{\gamma-1}}{s^{\gamma}(1-\gamma)+\gamma}. \tag{77}$$

Thus,

$$\frac{AB(\gamma)s^{\gamma}}{s^{\gamma}(1-\gamma)+\gamma}\mathscr{L}(\mathfrak{f}(\eta))=\beta\Gamma(2+\beta)s^{-2-\beta}. \tag{78}$$

Then,

$$\mathscr{L}(\mathfrak{f}(\eta))=\frac{\beta\Gamma(2+\beta)s^{-2-\beta}(s^{\gamma}(1-\gamma)+\gamma)}{AB(\gamma)s^{\gamma}}. \tag{79}$$

After applying the inverse Laplace transform, we will get the ES of (67) as

$$\mathfrak{f}(\eta)=\frac{1}{AB(\gamma)}\left[\beta\eta^{\beta+1}\left(1-\gamma+\frac{\gamma\eta^{\gamma}\Gamma(2+\beta)}{\Gamma(2+\gamma+\beta)}\right)\right]. \tag{80}$$

Using RKHSM, we can obtain several interesting results in which $\eta_{\mathfrak{i}}=\frac{\mathfrak{i}}{n}$, $\mathfrak{i}=1,2,\ldots,n$. Tables 6–10 show the ES, AS, AE, and RE of (67) at various $\eta\in[0,1]$ when $(\gamma;\beta)\in\{(0.1;1),(0.25;1),(0.5;0.77),(0.8;0.6),(0.9;0.5)\}$. Also, the comparaison between the ESs and ASs of this problem, which are obtained by (80) and RKHSM respectively, is depicted in Fig. 2 for different value of γ and β. From this figure, we can observe that the graphs are in good agreement with each other and they are also almost much, which confirms the efficiency of the RKHSM. Moreover, one can conclude that the RKHSM's solution is stable and convergent.

Problem 3. We take into consideration the following FFDE with the Mittag–Leffler kernel:

Table 6 The ES, AS, AE, and RE of Problem 2 at $\gamma = 0.9$ and $\beta = 0.5$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0025259721	0.002525970	1.8×10^{-9}	$7.122010575 \times 10^{-7}$
0.2	0.0092213972	0.009221394	3.2×10^{-9}	$3.490794208 \times 10^{-7}$
0.3	0.0205612685	0.020561265	3.5×10^{-9}	$1.682775558 \times 10^{-7}$
0.4	0.0370438314	0.037043823	8.4×10^{-9}	$2.272983025 \times 10^{-7}$
0.5	0.0591118844	0.059111868	1.6×10^{-8}	$2.765941262 \times 10^{-7}$
0.6	0.0871628721	0.087162848	2.4×10^{-8}	$2.764938720 \times 10^{-7}$
0.7	0.1215586089	0.121558577	3.2×10^{-8}	$2.624248524 \times 10^{-7}$
0.8	0.1626324067	0.162632371	3.6×10^{-8}	$2.195134458 \times 10^{-7}$
0.9	0.2106942543	0.210694190	6.4×10^{-8}	$3.051815543 \times 10^{-7}$
1	0.2660346634	0.266034690	2.7×10^{-8}	$9.998697034 \times 10^{-8}$

Table 7 The ES, AS, AE, and RE of Problem 2 at $\gamma = 0.8$ and $\beta = 0.6$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0044306331	0.004430630	3.1×10^{-9}	$6.886149146 \times 10^{-7}$
0.2	0.0157517652	0.015751761	4.2×10^{-9}	$2.634625365 \times 10^{-7}$
0.3	0.0341317328	0.034131727	5.8×10^{-9}	$1.711017729 \times 10^{-7}$
0.4	0.0599986089	0.059998597	1.2×10^{-8}	$1.976712498 \times 10^{-7}$
0.5	0.0937796203	0.093779598	2.2×10^{-8}	$2.376849035 \times 10^{-7}$
0.6	0.1358789440	0.135878913	3.1×10^{-8}	$2.281442517 \times 10^{-7}$
0.7	0.1866761091	0.186676060	4.9×10^{-8}	$2.630224094 \times 10^{-7}$
0.8	0.2465280847	0.246528040	4.5×10^{-8}	$1.813180841 \times 10^{-7}$
0.9	0.3157718890	0.315771820	6.9×10^{-8}	$2.185121678 \times 10^{-7}$
1	0.3947270039	0.394727050	4.6×10^{-8}	$1.167895775 \times 10^{-7}$

Table 8 The ES, AS, AE, and RE of Problem 2 at $\gamma = 0.5$ and $\beta = 0.77$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0100212389	0.010021235	3.9×10^{-9}	$3.891734384 \times 10^{-7}$
0.2	0.0365247847	0.036524778	6.7×10^{-9}	$1.828895107 \times 10^{-7}$
0.3	0.0785545287	0.078554520	8.7×10^{-9}	$1.103691938 \times 10^{-7}$
0.4	0.1358891199	0.135889099	2.1×10^{-8}	$1.538018645 \times 10^{-7}$
0.5	0.2084756998	0.208475670	3.0×10^{-8}	$1.429423191 \times 10^{-7}$
0.6	0.2963297648	0.296329710	5.5×10^{-8}	$1.849291111 \times 10^{-7}$
0.7	0.3995001858	0.399500120	6.6×10^{-8}	$1.647058058 \times 10^{-7}$
0.8	0.5180536997	0.518053630	7.0×10^{-8}	$1.345420369 \times 10^{-7}$
0.9	0.6520670380	0.652066920	1.2×10^{-7}	$1.809629887 \times 10^{-7}$
1	0.8016225679	0.801622630	6.2×10^{-8}	$7.746787888 \times 10^{-8}$

$$ {}_{FFM}^{\;\;0}D^{\eta}_{\gamma,\beta}\mathfrak{f}(\eta) = \sin(\eta), \qquad 0 \leqslant \eta \le 1, \quad 0 < \gamma, \beta \le 1. \tag{81} $$

The exact solution to the above problem is [9]

$$ \mathfrak{f}(\eta) = \frac{1}{AB(\gamma)} \int_0^{\eta} \beta\zeta^{\beta-1} \times \sin(\zeta)\left(\delta(\eta-\zeta) + \gamma\delta(\eta-\zeta) + \frac{\gamma(\eta-\zeta)^{\gamma-1}}{\Gamma(\alpha)}\right)\mathrm{d}\zeta, \tag{82} $$

where δ is the Dirac delta function.

We need first to convert problem (81) into FDE with ABC derivative before applying the RKHSM. Therefore, let us follow the same steps as used in Section 4:

We have

$$ \frac{AB(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\eta^{\beta}} \int_0^{\eta} \mathfrak{f}(\zeta)E_{\gamma}\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^{\gamma}\right)\mathrm{d}\zeta = \sin(\eta). \tag{83} $$

Then, we get

Table 9 The ES, AS, AE, and RE of Problem 2 at $\gamma = 0.25$ and $\beta = 1$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0105048053	0.010504799	6.3×10^{-9}	$5.997255370 \times 10^{-7}$
0.2	0.0430385033	0.043038491	1.2×10^{-8}	$2.853259074 \times 10^{-7}$
0.3	0.0983743895	0.098374370	2.0×10^{-8}	$1.980190180 \times 10^{-7}$
0.4	0.1770025630	0.177002530	3.3×10^{-8}	$1.864379783 \times 10^{-7}$
0.5	0.2792982888	0.279298220	6.9×10^{-8}	$2.463316202 \times 10^{-7}$
0.6	0.4055706137	0.405570530	8.4×10^{-8}	$2.063758990 \times 10^{-7}$
0.7	0.5560846634	0.556084540	1.2×10^{-7}	$2.219086555 \times 10^{-7}$
0.8	0.7310739725	0.731073860	1.1×10^{-7}	$1.538831968 \times 10^{-7}$
0.9	0.9307481013	0.930747910	1.9×10^{-7}	$2.055335915 \times 10^{-7}$
1	1.1552977090	1.155297810	1.0×10^{-7}	$8.742335349 \times 10^{-8}$

Table 10 The ES, AS, AE, and RE of Problem 2 at $\gamma = 0.1$ and $\beta = 1$ using 10 points.

η	ES	AS	AE	RE
0.1	0.0106785032	0.010678495	8.2×10^{-9}	$7.632155823 \times 10^{-7}$
0.2	0.0429419502	0.042941937	1.3×10^{-8}	$3.064602318 \times 10^{-7}$
0.3	0.0969362900	0.096936269	2.1×10^{-8}	$2.167402940 \times 10^{-7}$
0.4	0.1727449902	0.172744960	3.0×10^{-8}	$1.748241727 \times 10^{-7}$
0.5	0.2704285370	0.270428480	5.7×10^{-8}	$2.107765720 \times 10^{-7}$
0.6	0.3900348176	0.390034730	8.8×10^{-8}	$2.245953337 \times 10^{-7}$
0.7	0.5316036607	0.531603540	1.2×10^{-7}	$2.270488503 \times 10^{-7}$
0.8	0.6951692646	0.695169150	1.2×10^{-7}	$1.648519373 \times 10^{-7}$
0.9	0.8807616567	0.880761470	1.9×10^{-7}	$2.119756220 \times 10^{-7}$
1	1.0884076420	1.088407710	6.8×10^{-8}	$6.247659184 \times 10^{-8}$

$$\frac{AB(\gamma)}{1-\gamma}\frac{1}{\beta\eta^{\beta-1}}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^\eta \mathfrak{f}(\zeta)E_\gamma\left(-\frac{\gamma}{1-\gamma}(\eta-\zeta)^\gamma\right)\mathrm{d}\zeta = \sin(\eta). \tag{84}$$

Therefore, we reach

$${}^{0}_{ABR}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \beta\eta^{\beta-1}\sin(\eta). \tag{85}$$

Thanks to (10), we find

$${}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \beta\eta^{\beta-1}\sin(\eta) - \frac{AB(\gamma)}{1-\gamma}\mathfrak{f}(0)E_\gamma\left(-\frac{\gamma}{1-\gamma}\eta^\gamma\right). \tag{86}$$

Then, we obtain

$$\begin{cases} {}^{0}_{ABC}D^{\eta}_{\gamma}\mathfrak{f}(\eta) = \beta\eta^{\beta-1}\sin(\eta), \quad 0<\eta\le 1, \quad 0<\gamma<1, \\ \mathfrak{f}(0)=0. \end{cases} \tag{87}$$

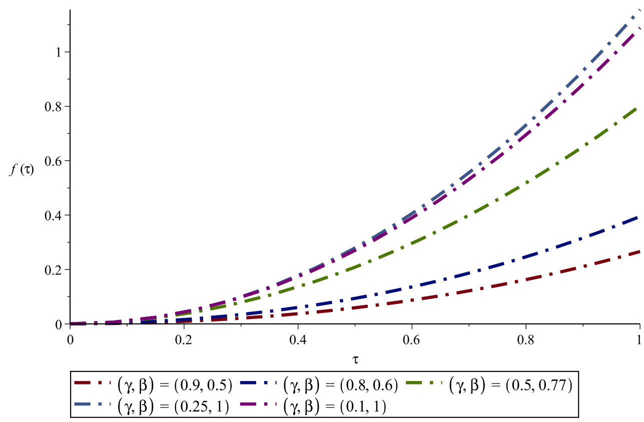
Put $\mathfrak{f}_n(\eta_1) = \mathfrak{f}(\eta_1) = 0$, the n-term solution's representation of (87) takes the form:

$$\mathfrak{f}_n(\eta) = \sum_{\mathfrak{i}=1}^{n}\sum_{k=1}^{\mathfrak{i}}\varpi_{\mathfrak{i}k}\Theta(\eta_k,\mathfrak{f}_{k-1}(\eta_k))\bar{\psi}_{\mathfrak{i}}(\eta), \tag{88}$$

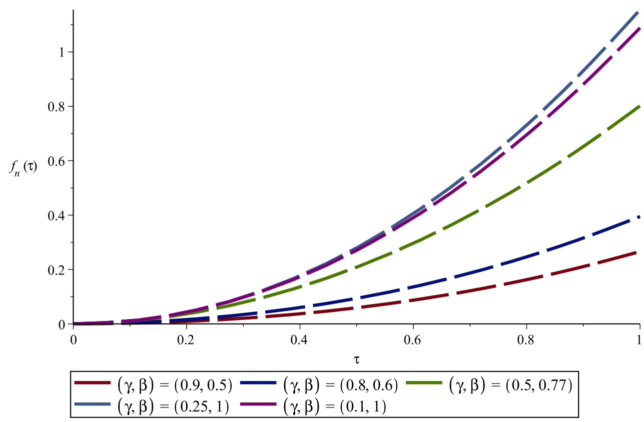
where $\Theta(\eta_k,\mathfrak{f}_{k-1}(\eta_k)) = \beta\eta_k^{\beta-1}\sin(\eta_k)$.

$$\begin{aligned} \mathfrak{f}_1(\eta) &= \varpi_{11}\Theta(\eta_1,\mathfrak{f}_0(\eta_1))\bar{\psi}_1(\eta), \\ \mathfrak{f}_2(\eta) &= \mathfrak{f}_1(\eta) + (\varpi_{21}\Theta(\eta_1,\mathfrak{f}_0(\eta_1)) + \varpi_{22}\Theta(\eta_2,\mathfrak{f}_1(\eta_2)))\bar{\psi}_2(\eta), \\ &\vdots \\ \mathfrak{f}_n(\eta) &= \mathfrak{f}_{n-1}(\eta) + \sum_{k=1}^{n}\varpi_{nk}\Theta(\eta_k,\mathfrak{f}_{k-1}(\eta_k))\bar{\psi}_n(\eta). \end{aligned}$$

Using RKHSM, we can obtain several interesting results in which $\eta_{\mathfrak{i}} = \frac{\mathfrak{i}}{n}$, $\mathfrak{i} = 1,2,\ldots,n$ and $n = 10$. Figs. 3 and 4 show the numerical results of this problem which are obtained by (82) and RKHSM for various values of γ and β. In Fig. 4, η is moving in $[0,1]$ and $(\gamma;\beta) \in \{(0.1;1),(0.25;1),(0.5;0.85),(0.8;0.9),(0.9;0.9)\}$. Whilst in Fig. 3, η and γ are moving in $[0,1]$ when β is fixed at 1. This last figure give as that the numerical solution is continuously depends on the $\gamma\eta-$ levels surface when $\beta = 1$. We can observe, from these figures, that the graphs are in good agreement with each other and they are also almost much, which confirms the efficiency of the RKHSM. Moreover, one can conclude that the RKHSM's solution is stable and convergent.

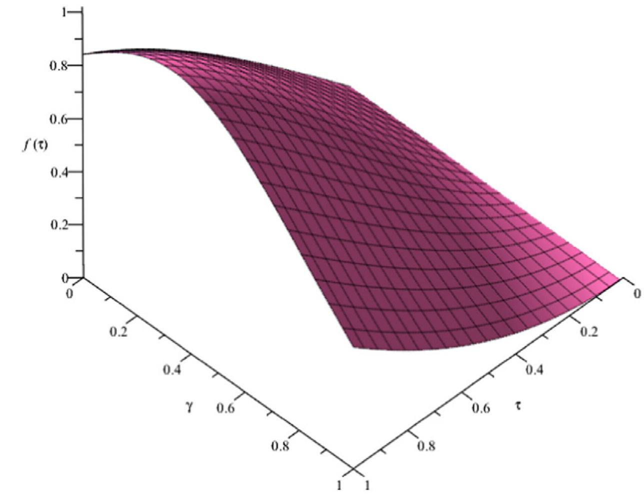


(a) ESs for different values of γ and β

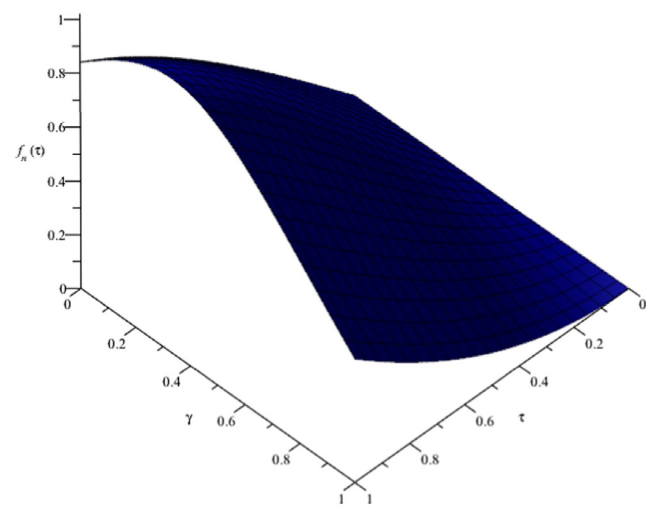


(b) ASs for different values of γ and β

Fig. 2 Comparison between the exact and RKHSM's solutions for Problem 2 at various values of γ and β.



(a) ES when $\eta \in [0, 1]$ for γ is moving in $[0, 1]$ and $\beta = 1$

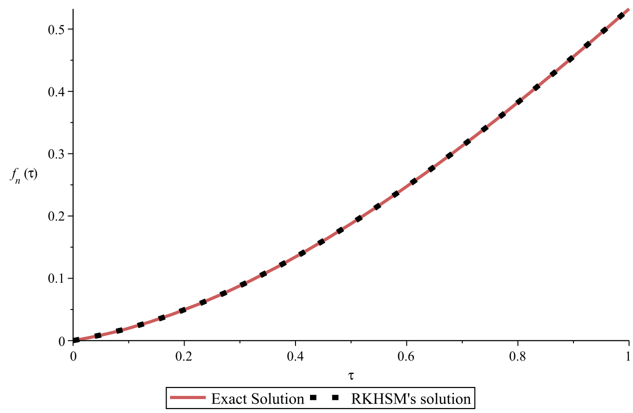


(b) AS when $\eta \in [0, 1]$ for γ is moving in $[0, 1]$ and $\beta = 1$

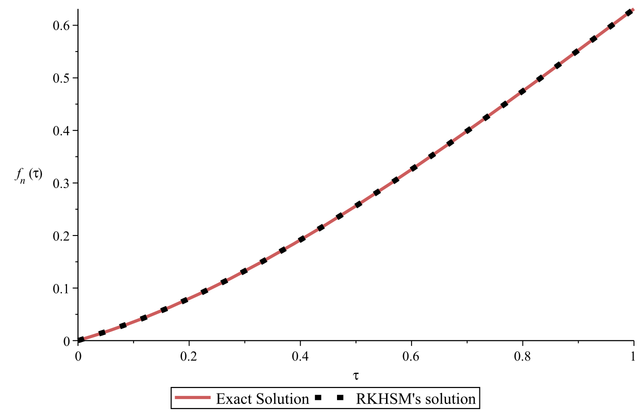
Fig. 3 Comparison between the exact and RKHSM's solutions for Problem 3 at various values of γ and for $\beta = 1$.

7. Conclusion

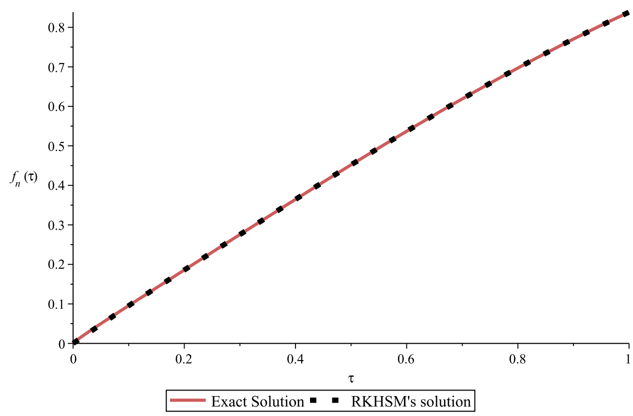
In the present work, we succeed in developing an appropriate expansion for solving some important fractal-fractional ODEs with the generalized M-L kernel. We offered appropriate series solutions for this new class of equations by using the RKHSM. The numerical solutions $\mathfrak{f}_n(\eta)$ and their derivatives converge uniformly. The used differential operator is bounded and linear. The RKHSM is shown to have good convergence. The comparisons between the results obtained by using the exact solution (The is derived via the Laplace transform) and the RKHSM indicates that the solutions' results are good agreement with each other, the Laplace transform is a very effective tool to derive the exact solution, and the RKHSM is pretty accurate. We observed that the RKHSM is a suitable, easy, and effective way to investigate this new class of ODEs. We can also use the RKHSM to solve other kinds of problems, such as non-linear PDEs, which will be considered in the future.



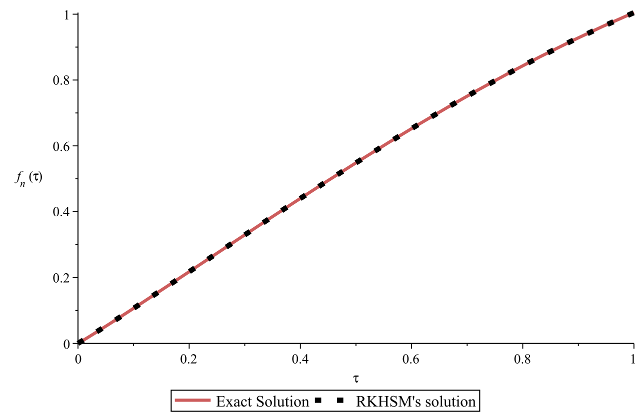
(a) $(\gamma;\beta) = (0.9;0.9)$



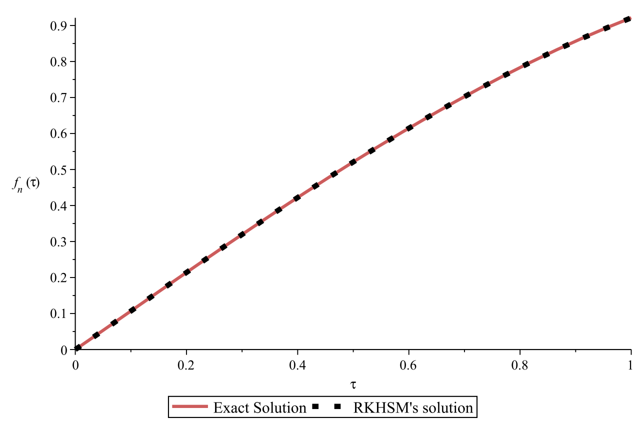
(b) $(\gamma;\beta) = (0.8;0.9)$



(c) $(\gamma;\beta) = (0.5;0.85)$



(d) $(\gamma;\beta) = (0.25;1)$



(e) $(\gamma;\beta) = (0.1;1)$

Fig. 4 Comparison between the exact and RKHSM's solutions for Problem 3 at various values of γ and β.

Data availability

Data are included within this research.

Author contribution

Nourhane Attia: Data curation, Writing – original draft. **Ali Akgül:** Conceptualization, Methodology, Software, Validation. **Djamila Seba:** Visualization, Investigation. **Abdelkader Nour and Jihad Asad:** Supervision.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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